The principle of minimum potential energy naturally leads to the same system, since $\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} = \boldsymbol{0}$ guarantees $\delta\Pi(\boldsymbol{u}) = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} = \boldsymbol{0}$, i.e. $\Pi(\boldsymbol{u})$ has a horizontal tangent at the lowest point, and this in each direction u_i.

9.10 Weak Solution

Given the boundary value problem of a rope

$$-H\, w''(x) = p(x) \qquad w(0) = w(l) = 0\,, \tag{9.69}$$

we determine the FE-solution $w_h(x) = \sum_j w_j\, \varphi_j(x)$ by requiring

$$\int_0^l H\, w_h'\, \varphi_i'\, dx = \int_0^l p\, \varphi_i\, \varphi_i\, dx \qquad \text{for all } \varphi_i \in \mathcal{V}_h\,. \tag{9.70}$$

The solution of the boundary value problem (9.69) is called a ***strong solution*** and the solution of the variation problem (9.70) is called a ***weak solution***.

This distinction is usually explained by saying a weak solution does not have to be twice differentiable, as in $-H\, w'' = p$, but only once, as the w' in the strain energy product.

The following interpretation, though, seems more appropriate. In mathematics there is the concept of ***weak convergence***. It is an indirect proof for a sequence of functions $f_n(x)$ to converge to a target function $f(x)$. The test $f_n(x) \to f(x)$ is done against a set of control functions $\varphi_i(x)$, and the sequence $f_n(x)$ is said to converge *weakly* to $f(x)$ if

$$\lim_{n\to\infty} \int_0^l f_n(x)\, \varphi_i(x)\, dx = \int_0^l f(x)\, \varphi_i(x)\, dx \qquad \text{for all } \varphi_i\,. \tag{9.71}$$

Weak convergence is like the convergence of functionals. Every function $f_n(x)$ can be equated to a functional $J_n(.)$

$$J_n(\varphi_i) = \int_0^l f_n(x)\, \varphi_i(x)\, dx\,, \tag{9.72}$$

and weak convergence means, the functionals $J_n(.)$ converge towards the **target functional**

$$J(\varphi_i) = \int_0^l f(x)\, \varphi_i(x)\, dx\,, \tag{9.73}$$

in the sense of (9.71), the limit of the sequence $J_n(.)$ is “shake equivalent” to $J(.)$.

And this terminology fits exactly. The FE-solution is a weak solution since its agreement with the exact solution is not controlled by the differential equation, but it is controlled indirectly by $i = 1, 2, \ldots$ "shake tests"

$$\lim_{h \to 0} a(u_h, \varphi_i) = a(u, \varphi_i) \qquad \text{for all } \varphi_i \,. \tag{9.74}$$

Because of $\delta W_i = \delta W_e$ this is identical to

$$\lim_{h \to 0} \int_0^l p_h \, \varphi_i \, dx = \int_0^l p \, \varphi_i \, \varphi_i \, dx \qquad \text{for all } \varphi_i \tag{9.75}$$

i.e. the equivalence of the external virtual work. In practice, of course, the finite elements never reach the limit $h \to 0$ and only a finite number of tests is performed, since there are only a finite number of shape functions φ_i on a mesh.

The finite element method can be seen as a method of replacing a functional $J(\delta u) = (p, \delta u)$ with a functional $J_h(\delta u) = (p_h, \delta u)$, or, if you have infinite patience, $h \to 0$, with a sequence of functionals $J_h(\delta u) = (p_h, \delta u)$.

The practical significance of the concept of a weak solution becomes apparent when we watch a market woman, see Fig. 9.5, since she also draws conclusions indirectly. She has to solve the equation

$$P_l \cdot h_l = P_r \cdot h_r \,, \tag{9.76}$$

which means, as she knows, that at every turn $\delta\varphi$ of the balance beam, the work on the left and right side are the same

$$P_l \cdot h_l = P_r \cdot h_r \qquad \Rightarrow \qquad P_l \cdot h_l \cdot \tan \tan \delta\varphi = P_r \cdot h_r \cdot \tan \delta\varphi \,, \tag{9.77}$$

and so she concludes, by wiggling the balance, indirectly, arguing "backwards"

$$P_l \cdot h_l = P_r \cdot h_r \qquad \Leftarrow \qquad P_l \cdot h_l \cdot \tan \delta\varphi = P_r \cdot h_r \cdot \tan \delta\varphi \,. \tag{9.78}$$

The same is done by the toolmaker, who rolls a cylinder back and forth with his fingers, see Fig. 9.6, since he knows: If the cylinder has a perfect circular shape, the center of gravity does not change its height above the table when the cylinder is rotated. If the test fails, if his fingers feel a slight wobble, it is not a perfect cylinder.

The apprentice who is supposed to grind a cylinder out of a square iron, does it like the finite elements. At the start the square iron is equivalent to a cylinder with respect to all rotations $\delta\varphi$, which are a multiple of 90°, the center of gravity does not change its height. By grinding more and more edges (n) into the profile, the apprentice increases the test space, $\mathcal{V}_4 \to \mathcal{V}_8 \to \mathcal{V}_{16} \ldots$

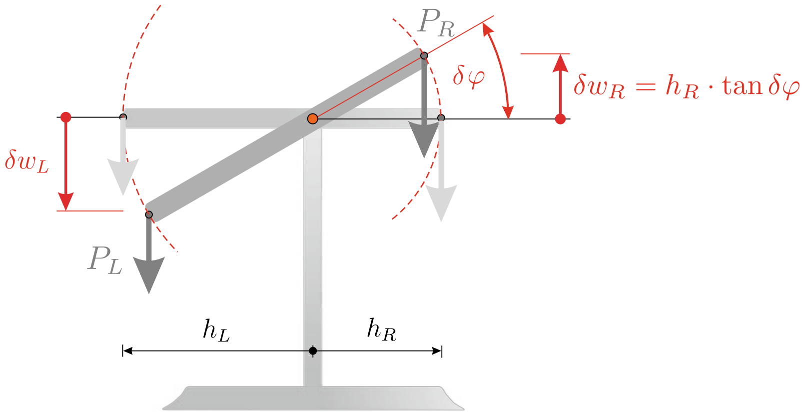

Fig. 9.5 The market woman checks the equilibrium with the principle of virtual displacements

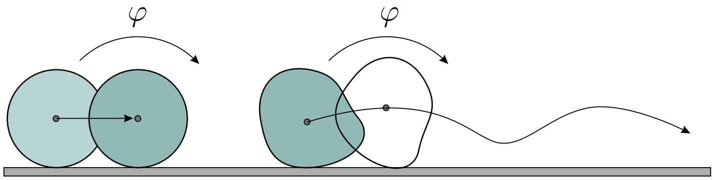

Fig. 9.6 The tool maker checks the excentricity by rolling the cylinder across the table

$$\mathcal{V}_n = \{\text{all multiples of } \delta\varphi = \frac{360}{n}\} \qquad n = 4, 8, 16 \ldots \text{edges} \tag{9.79}$$

and he so approximates the shape indirectly (via rotational equivalence), see Fig. 9.7 [2].

Equivalence is the key term in FE-analysis. The FEM does not solve the original load case but an equivalent load case. An equivalence relation means if $a \sim b$ and $b \sim c$ also $a \sim c$, and so

$$p \sim \varphi_i \qquad \text{and} \qquad p_h \sim \varphi_i \qquad \Rightarrow \qquad p \sim p_h\,. \tag{9.80}$$

In the finite element method this equivalence is "finite", is established only with respect to a finite set of test functions φ_i, $i = 1, 2, \ldots n$.

If we compare the length of two boards A and B via a folding rule, we use an equivalence relation. The boards have the same length, are equivalent, if they are in identical relations with the folding rule. Equivalence is indirect equality, is like weak convergence, and it leads to a true identity, $A \equiv B$, (all places after the decimal point are equal), if the relation passes all tests, and also the test with the Urmeter in Paris...

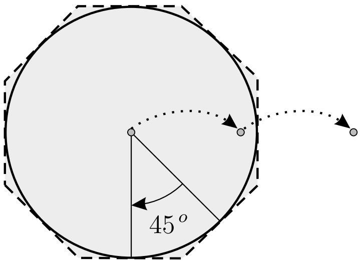

Fig. 9.7 The octogon is equivalent to a cylinder with respect to all rotations, which are a multiple of 45°

Remark 9.1 The concept of weak convergence also includes the difference between ***weak*** and ***strong boundary conditions***. Geometric boundary conditions are strong boundary conditions, like $w = 0$, since they are satisfied by all shape functions $\varphi_i \in \mathcal{V}_h$, while a static boundary condition like $v_n = 0$ (zero Kirchhoff shear) at a free edge of a plate, is guaranteed only in the integral mean—in the weak sense—$(v_n, \varphi_i) = 0$, but not pointwise. This is why static boundary conditions are called weak boundary conditions.

9.11 Variation and Green's First Identity

The potential energy of a bar, held fixed on the left, $u(0) = 0$, with a free end, $N(l) = 0$, and carrying a load p is

$$\Pi(u) = \frac{1}{2}\int_0^l \frac{N^2}{EA}\,dx - \int_0^l p\,u\,dx = \frac{1}{2}a(u,u) - (p,u)\,. \tag{9.81}$$

In the equilibrium position u the first variation of $\Pi(u)$ should be zero.

The value of Π at a neighboring point $u + \varepsilon\,\delta u$ is

$$\begin{aligned}\Pi(u+\varepsilon\delta u) &= \frac{1}{2}a(u+\varepsilon\delta u, u+\varepsilon\delta u) - (p, u+\varepsilon\delta u)\\ &= \frac{1}{2}a(u,u) + \varepsilon\cdot a(u,\delta u) + \varepsilon^2\cdot\frac{1}{2}a(\delta u,\delta u) - (p,u) - \varepsilon\cdot(p,\delta u)\end{aligned} \tag{9.82}$$

and so the first variation

$$\delta\Pi(u,\delta u) = \frac{d}{d\varepsilon}\Pi(u+\varepsilon\delta u)|_{\varepsilon=0} = a(u,\delta u) - (p,\delta u) \tag{9.83}$$

is identical to

$$\mathscr{G}(u, \delta u) = \int_0^l p\, \delta u(x)\, dx - \int_0^l \frac{N\, \delta N}{EA}\, dx = (p, \delta u) - a(u, \delta u) = 0 \quad (9.84)$$

since the work done on the boundary [...] is zero because of $N(l) = 0$ and $\delta u(0) = 0$.

The first variation of the potential energy is Green's first identity.

9.12 The Basic Functional (Hu-Washizu)

In [3] we introduced the basic functional Π_H and we showed how it can be derived from Green's second identity. We call it basic since all other functionals are modifications of this functional, in particular the **potential energy** and the **complementary potential energy**. We want to use this result to show that the potential energy, $\Pi(u) = 1/2\, a(u, u)$, is positive if a support settles.

To keep the algebra simple, we consider a bar. By integrating the last integral in Green's second identity by part

$$\mathscr{B}(u, \delta u) = \int_0^l -EAu''\delta u\, dx + [N\, \delta u - u\, \delta N]_0^l - \underbrace{\int_0^l u(-EA\delta u'')\, dx}_{int.\,by\,parts} = 0 \quad (9.85)$$

we can make a step back in the direction of $a(u, \delta u)$, and we thus obtain

$$\mathscr{V}(u, \delta u) = \left\{ \int_0^l -EAu''\delta u\, dx + [N\, \delta u - u\, \delta N]_0^l \right\} + [u\delta N]_0^l - a(u, \delta u) = 0\,. \quad (9.86)$$

The curly brackets contain the "rest" of Betti, the untouched part.

Next, we replace all terms of u inside the curly brackets with their data as far as they appear in the boundary value problem, remove the brackets and delete all terms which cancel. The resulting expression is the first variation of the **basic functional** Π_H (H as in *Hu-Washizu*).

Consider for example the boundary value problem

$$-EAu'' = p \qquad u(0) = 0 \qquad u(l) = \Delta\,. \quad (9.87)$$

Applying the above scheme we get

$$\mathscr{V}(u, \delta u) = \int_0^l p\, \delta u\, dx + [N\delta u]_0^l - \Delta\, \delta N(l) + [u\, \delta N]_0^l - a(u, \delta u) = 0\,, \quad (9.88)$$