It is only, if we let $\boldsymbol{f} = \boldsymbol{K}\boldsymbol{u}$, that it looks like a weak formulation

$$N_h(x) = \boldsymbol{g}^T \boldsymbol{f} = \boldsymbol{g}^T \boldsymbol{K}\boldsymbol{u}\,. \tag{9.142}$$

9.23 How the Embedding Theorem Got Its Name

In the same way as the citizens of a city can be classified according to different criteria, *age, size*, ..., so the deflections $w(\boldsymbol{x})$ of a plate Ω can be classified in different ways. One of these possible scales is the so-called ***Sobolev norm***. The Sobolev norm of order $m = 2$ of a function $w(x, y)$ is the number

$$\|w\|_2 = \sqrt{\int_\Omega (w^2 + w,_x^2 + w,_y^2 + w,_{xx}^2 + w,_{xy}^2 + w,_{yy}^2)\, d\Omega}\,. \tag{9.143}$$

Following this pattern, we can define Sobolev norms $\|w\|_m$ to arbitrarily high order m: Square integrate the derivatives up to the order m and take the square root of the sum! The functions with a finite Sobolev norm of order m form the **Sobolev space** $H^m(\Omega)$.

Closely related to Sobolev spaces are the so-called ***energy spaces*** of functions with finite strain energy, $a(w, w) < \infty$, since the energy space of a Kirchhoff plate can be identified with the Sobolev space H^2, and the energy space of a plate with $\boldsymbol{H}^1 = H^1 \times H^1$ (horizontal displ. u_x and vertical displ. u_y).[3]

The Russian mathematician S. Sobolev has shown that the space $H^m(\Omega)$ is embedded into the space $C(\Omega)$ of the continuous functions over Ω,

$$H^m(\Omega) \subset C(\Omega)\,, \tag{9.144}$$

and the **embedding** is even continuous

$$\max|w| < c \cdot \|w\|_m\,, \tag{9.145}$$

if

$$m > \frac{n}{2}\,, \tag{9.146}$$

if the index m of the Sobolev space is greater than half the dimension of the space, which would be $n = 2$ for a Kirchhoff plate. The value $\max|w|$ is the norm of a function w in $C(\Omega)$, and (9.145) means: The norm $|w|$ of the projection is smaller

[3]Technically, the energy norm $\|w\|_E = \sqrt{a(w, w)}$ and the Sobolev norm $\|w\|_m$ are equivalent norms, if the rigid-body motions are excluded.

than the initial norm $\|w\|_m$ times a constant factor c, which only depends on the shape of Ω.

A small distance between two functions w_1 and w_2 in $H^m(\Omega)$ remains a small distance in $C(\Omega)$

$$\max |w_1 - w_2| < c \cdot \|w_1 - w_2\|_m \,, \tag{9.147}$$

—the neighborhood is preserved.

The interesting aspect of this result is the transition from an integral to a point. The Sobolev norm is an integral measure, but if the inequality (9.146) applies, as in the case of a Kirchhoff plate, $m = 2$, $n = 2$, the maximum deflection of the plate is bounded by the strain energy. If the strain energy tends to zero, the deflection of the plate also tends to zero—everywhere, at each point! ***An integral majorizes point values!***

In the case of a wall plate the situation is different. The energy space is $\boldsymbol{H}^1$, but since $m = 1$ and $n = 2$ do not satisfy the inequality (9.146), a small distance $\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1$ between two displacement fields does not guarantee the maximum displacements to be equally close, which for a Kirchhoff plate is true

$$\max |w_1 - w_2| < c \cdot \|w_1 - w_2\|_2 \,. \tag{9.148}$$

If the reader has some patience left, we can pursue these ideas a little further.

People can be classified with respect to age (A) or weight (W). On the set A the functional "shoe size" is not constant, since a small distance in the age of two persons is no guarantee for similar shoe sizes. On the other hand, we (approximately) expect such a correlation to exist on the set W.

So, a functional is continuous if a small difference in the input guarantees a small distance in the output, if

$$|J(u_1) - J(u_2)| < c \cdot \|u_1 - u_2\| \,, \tag{9.149}$$

where c is a global constant, independent of the single u. Continuity therefore depends on how one measures distance in the input set and in the target set. Note, if you set $u_2 = 0$ such a bound implies $|J(u)| < c \cdot \|u\|$.

If a linear functional is continuous, it is also bounded and vice versa.

The point functional

$$J(w) = w(\boldsymbol{x}) \qquad \text{deflection at a point } \boldsymbol{x} \tag{9.150}$$

of a Kirchhoff plate is a continuous and bounded functional on $H^2(\Omega)$, because the continuous embedding (9.145) guarantees that the difference in two values is bounded by the difference in their energies

$$|J(w_1) - J(w_2)| = |w_1(\boldsymbol{x}) - w_2(\boldsymbol{x})| < c \cdot \|w_1 - w_2\|_2\,, \tag{9.151}$$

but this does not hold true for wall plates, since it cannot be guaranteed *for all (!)* $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ that the difference in the horizontal displacement of two displacement fields at a point $\boldsymbol{x}$

$$|J(\boldsymbol{u}) - J(\boldsymbol{v})| = |u_x(\boldsymbol{x}) - v_x(\boldsymbol{x})| \not< c \cdot \|\boldsymbol{u} - \boldsymbol{v}\|_1 \tag{9.152}$$

is bounded by the Sobolev norm $\|\boldsymbol{u} - \boldsymbol{v}\|_1$.

The emphasis is on *for all*. The deflection $u = -\ln(-\ln^{-1} r)$ of a membrane is infinite at the point $r = 0$, $J(u) = \infty$, but in a disk Ω with radius $R = 0.5$ and centered at the origin the H^1 norm is bounded, see [7], p. 98. So, there exists no bound c

$$J(u) = u(\boldsymbol{x}) = \infty < c \cdot \|u\|_1 \qquad ? \tag{9.153}$$

and therefore $J(u)$ is not a continuous and bounded functional on $H^1(\Omega)$. One counterexample is enough to substantiate this claim.

Given a bar, $n = 1$, $m = 1$, the energy space is $H^1(0, l)$, and this means

$$J(u) = u(x) \qquad m - i = 1 - 0 = 1 > \frac{1}{2} \qquad \text{continuous} \tag{9.154}$$

$$J(u) = u'(x) \qquad m - i = 1 - 1 = 0 \not> \frac{1}{2} \qquad \text{not continuous} \tag{9.155}$$

The energy space of a beam, $n = 1$, $m = 2$, is $H^2(0, l)$, and so we have

$$\underbrace{J(w) = w(x) \qquad J(w) = w'(x)}_{continuous} \qquad \underbrace{J(w) = w''(x) \qquad J(w) = w'''(x)}_{not\ continuous}\,. \tag{9.156}$$

If a functional is continuous, the energy of the Green's function is finite, otherwise it is infinite and also the external work $W_e = displ. \times force$ is infinite. One of the two quantities must be infinite at the source point.

The Green's function of the (discontinuous) functional $J(u) = EA\,u'(x)$, the normal force $N(x)$ in a bar, is a dislocation and to produce such a gap requires infinitely large forces, which let the strain energy "overflow"

$$\int_0^l \frac{N^2}{EA}\,dx = \infty \qquad N = \text{normal force due to dislocation}\,. \tag{9.157}$$

Here, we follow the mathematician, who expands the dislocation into a Fourier series, see page 73, and not the engineer, who first installs a normal force hinge and then spreads the hinge.

Next, a comment on the inequality (1.268) from Chap. 1, which we repeat here

$$m - i > \frac{n}{2}\,. \tag{9.158}$$

If we differentiate a function in $H^m(\Omega)$, its derivative (possibly) lies one order of space lower, lies in $H^{m-1}(\Omega)$. This explains why we subtract from m the index $i = 0, 1, 2, 3$ of the Dirac delta δ_i, i.e. the signal how often the Dirac delta differentiates the function u

$$J(u) = \int_\Omega \delta_i(\boldsymbol{y} - \boldsymbol{x})\, u(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} \quad \sim \quad \frac{d^i u}{dx^i}\,. \tag{9.159}$$

Remark 9.2 Not continuous in (9.156) means: $J(w) = -EI\, w''(x) = M(x)$ is not a continuous ***functional*** on $H^2(0, l)$. There exists no constant c so that *all* functions $w \in H^2(0, l)$ satisfy the inequality

$$|J(w)| = |M(x)| < c \cdot \|w\|_2\,. \tag{9.160}$$

A curve w with a logarithmic singularity in the bending moment, $M(x) = \ln(x - x_0)^2$ at a point $x_0 \in (0, l)$ has $\|w\|_2 < \infty$, but $M(x_0) = \infty$.

We can read it also this way: The singular $M(x)$ is in $H^0(0, l)$, because we can integrate $M(x)^2$, but the norm on H^0, the integral $\|M\|_0 = (M, M)^{1/2}$, is not a bound for $\max |M| < c \cdot \|M\|_0$, since the embedding of $H^0(0, l)$ into $C^0(0, l)$ is not continuous, there are outliers and one of these is the singular moment. Even simpler:

That we can square-integrate a function $M(x)$ on $(0, l)$, does not imply that the function $M(x)$ is bounded on the interval $(0, l)$.

But if also the derivative $M'(x)$ is square-integrable, if $M(x)$ lies in $H^1(0, l)$, the conclusion is correct, since then the inequality $m > n/2$, set $m = 1, n = 1$, is satisfied.

But the derivative $M'(x) = 2/(x - x_0)$ of $M(x) = \ln\,(x - x_0)^2$ is not square-integrable, and so the loophole $M \in H^1(0, l)$ is not available.

Remark 9.3 The mathematical theory of finite elements is based on the concept of weak solutions on appropriate Hilbert spaces. With regard to influence functions the rule is: *An influence function*

$$J(u) = \int_0^l G(y, x)\, p(y)\, dy \tag{9.161}$$

exists if and only if the functional $J(u)$ *is linear and bounded*, $|J(u)| < c\, \|u\|$.

If one follows this *dictum*, there should be no influence function $G_2(y, x)$ for the moment in a beam, since the functional $J(w) = M(x)$ is unbounded on $H^2(0, l)$.

However, an engineer would tell you otherwise, since she or he simply places a kink at the source point and so manages to generate the exact influence for the bending moment.

But the above statement only means "no influence function with ***finite energy***", no influence function $G_2(y, x) \in H^2(0, l)$, which, if we calculate, see page 73, is confirmed: The triangular shape with the kink has an infinite energy, does not lie in $H^2(0, l)$. Now, of course, one could say: If mathematicians are so subtle and draw such narrow bounds, the engineer need not care.

But this is a weakness in the mathematical theory of finite elements according to which the FEM is an ***energy method***, and solutions which have infinite energy lie outside this theory. How do you minimize the distance to the exact influence function G in the energy,

$$a(G - G_h, G - G_h) = \|G - G_h\|^2 \qquad \to \text{ minimum}\,, \tag{9.162}$$

if the energy of the solution, $\|G\| = \infty$, is infinite?

This results in a curious situation: An FE-program calculates all section forces with influence functions which are approximate solutions of *ill posed problems*, ill posed, because the exact solutions do not lie in the solution space, in the energy space—all influence functions for internal forces have infinite energy.

A situation which every mathematician warns against: "Please make sure a solution exists, before you start the computer..."

9.24 Point Loads and Their Energy

Table 9.1 evaluates the inequality $m - i > n/2$, which must be true for the strain energy to be finite; it is m the order of energy

$m = 1$ Timoshenko beam, Reissner–Mindlin plate, 2-D and 3-D elasticity

$m = 2$ Euler–Bernoulli beam, Kirchhoff plates

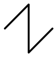
The order of the energy corresponds to the highest derivative in the strain energy $a(u, u)$. It is always half the order $2\,m$ of the differential equation.

Figure 9.12 illustrates why it is better to convert point loads into short line loads.

9.25 Early Birds

We know two early papers, which deal with Green's functions and finite elements, *Kolář* (Brno, Czech) [8], and *Tottenham* (Southampton) [9], both from 1970. There are probably other early papers. We would be grateful for hints.

Table 9.1 Finite (✓) and infinite (∞) energy

	dimension	$n=1$	$n=2$	$n=3$
	$m=1$	rope, bar, Timoshenko beam	2-D E-Th. Reissner–Mindlin plate	E-Th.
	point loads			
$i=0:$	↓	✓	∞	∞
$i=1:$		∞	∞	∞
	$m=2$	Euler–Bernoulli beam	Kirchhoff plate	
$i=0:$	↓	✓	✓	
$i=1:$	↷	✓	∞	
$i=2:$		∞	∞	
$i=3:$		∞	∞	

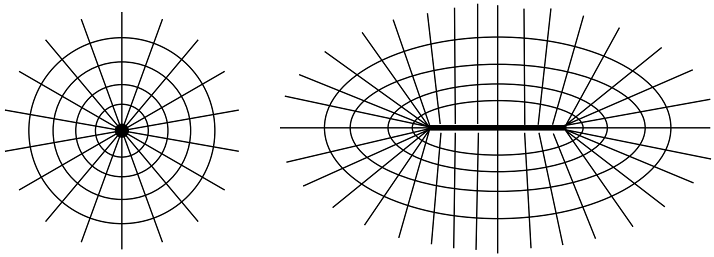


Fig. 9.12 Near the point load the lines of forces are so tightly packed that the material starts to flow, while a line load avoids the high stress concentration—the strain energy remains finite [7]

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