Chapter 8
Nonlinear Problems

In this chapter we discuss nonlinear problems. The focus is on the questions, which parts of the algebra of finite elements can be directly transferred to the nonlinear theory and where the nonlinear theory differs from the linear theory and what are the characteristic features of nonlinear theories.

To be as illustrative as possible, we have included selected examples to detail the numerical analysis of nonlinear problems.

The transition from linear theory to nonlinear theory, unfortunately, means a loss of transparency in the formulations, since nonlinear finite element formulations consist to a large part in a seemingly endless application of the product and chain rules of calculus, coordinate transformations, first-order approximations, and integration by parts and all this spiced with linear algebra. (For help with the technical issues see [1]).

We have tried to make the walk through the subject as transparent as possible.

8.1 Introduction

The key point in any nonlinear formulation are the three steps $u \to \varepsilon \to \sigma \to p$. If these are understood and formulated correctly Green's first identity

$$\mathscr{G}(u, \delta u) = a_u(u, \delta u) - (p, \delta u) = 0\,, \tag{8.1}$$

formulates itself automatically and the "rest" is just algebra and a fast computer.

As in the linear theory we let the FE-solution $u_h = \sum_j u_j\, \varphi_j(x)$ and we find the nodal displacements u_i by solving the n equations

$$a_u(u_h, \varphi_i) - (p, \varphi_i) = 0 \qquad i = 1, 2, \ldots, n\,, \tag{8.2}$$

F. Hartmann and P. Jahn, *Statics and Influence Functions*,
Springer Series in Solid and Structural Mechanics 13,
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or

$$\boldsymbol{k}(\boldsymbol{u}) = \boldsymbol{f} \tag{8.3}$$

for short, where $k_i = a_u(u_h, \varphi_i)$ and $f_i = (p, \varphi_i)$.

The vector $\boldsymbol{f}$ is the virtual work of the load on acting through the φ_i, and the left side is the increment of the virtual internal energy on acting through the φ_i. One moves from the equilibrium position $\boldsymbol{u}$ in a direction φ_i and checks whether the increase of the virtual internal energy corresponds to the increase of the virtual external work. This increment $a_u(u_h, \varphi_i)$ is the *Gateaux derivative* of the strain energy.

8.2 Gateaux Derivative

Let $J(u)$ be a (possibly nonlinear) functional, the expression

$$J_u(\delta u) = \frac{d}{d\varepsilon} J(u + \varepsilon \delta u)_{|\varepsilon=0} \tag{8.4}$$

is the Gateaux derivative of $J(u)$ in the direction of the increment δu.

We form $J(u + \varepsilon \delta u)$ with a test function δu (virtual displacement), differentiate with respect to ε, and set $\varepsilon = 0$ at the end.

This derivative (actually it is a differential, an increment) looks like a stopgap solution, if something cannot be differentiated correctly, one replaces it by a difference quotient, and one takes the limit.

Surprisingly, however, this derivative appears ***automatically*** in many nonlinear formulations, as for example in Green's identity of nonlinear elasticity, see (8.48),

$$\mathscr{G}(\boldsymbol{u}, \boldsymbol{\delta u}) = \int_\Omega \boldsymbol{p} \cdot \boldsymbol{\delta u}\, d\Omega + \int_{\Gamma_N} \bar{\boldsymbol{t}} \cdot \boldsymbol{\delta u}\, ds - \int_\Omega \boldsymbol{E}_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\delta u}) \cdot \boldsymbol{S}\, d\Omega = 0\,, \tag{8.5}$$

where

$$\boldsymbol{E}_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\delta u}) := \frac{1}{2}\,(\nabla \boldsymbol{\delta u} + \nabla \boldsymbol{\delta u}^T + \nabla \boldsymbol{u}^T\, \nabla \boldsymbol{\delta u} + \nabla \boldsymbol{\delta u}^T\, \nabla \boldsymbol{u}) \tag{8.6}$$

is the Gateaux derivative (= increment) of the nonlinear strain tensor $\boldsymbol{E}(\boldsymbol{u})$ at the point $\boldsymbol{u}$ in the direction of $\boldsymbol{\delta u}$,

$$\frac{d}{d\varepsilon} [\boldsymbol{E}(\boldsymbol{u} + \varepsilon \boldsymbol{\delta u})]_{|\varepsilon=0} = \boldsymbol{E}_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\delta u})\,. \tag{8.7}$$

So, what—at first glance—looks like a trick is an integral part of the variational formulations of nonlinear problems.

The symmetric strain energy product of linear elasticity

$$a(\boldsymbol{u}, \boldsymbol{\delta u}) = \int_\Omega \boldsymbol{E}(\boldsymbol{\delta u}) \boldsymbol{\cdot} \boldsymbol{S}(\boldsymbol{u})\, d\Omega = \int_\Omega \boldsymbol{E}(\boldsymbol{u}) \boldsymbol{\cdot} \boldsymbol{S}(\boldsymbol{\delta u})\, d\Omega = a(\boldsymbol{\delta u}, \boldsymbol{u})\,, \tag{8.8}$$

is replaced in the nonlinear theory by the integral

$$a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\delta u}) = \int_\Omega \boldsymbol{E}_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{\delta u}) \boldsymbol{\cdot} \boldsymbol{S}(\boldsymbol{u})\, d\Omega\,, \tag{8.9}$$

which is the scalar product of the Gateaux derivative of the strain tensor with the stress tensor. This integral is the increase in internal energy when $\boldsymbol{u}$ shifts to $\boldsymbol{u} + \boldsymbol{\delta u}$. In nonlinear problems Green's first identity is an "incremental view" of $\delta W_e - \delta W_i = 0$.

One could argue that also linear theory is an (exact) incremental view—exact because all expressions are linear—while the nonlinearity makes it truly incremental, a first-order approximation.

One consequence of the nonlinearity is that the simple algebra

$$\mathscr{B}(u, \hat{u}) = \mathscr{G}(u, \hat{u}) - \mathscr{G}(\hat{u}, u) = 0\,, \tag{8.10}$$

on which *Betti's theorem* is based is not available. In nonlinear problems there is no "Betti", symmetry is lost.

8.3 Nonlinear Bar

The governing equations are

strains	$\varepsilon - (u' + \frac{1}{2}\,(u')^2) = 0$	(8.11a)
const. equation	$\sigma - E\,\varepsilon = 0$	(8.11b)
equilibrium	$-A(\sigma + u'\sigma)' = p \qquad .$	(8.11c)

Integration by parts of the virtual work integral, we write $\mathscr{N} = A\,(\sigma + u'\sigma)$,

$$\int_0^l -\mathscr{N}'\,\delta u\, dx = -[\mathscr{N}\,\delta u]_0^l + \int_0^l \mathscr{N}\,\delta u'\, dx = 0\,, \tag{8.12}$$

where

$$\mathscr{N}\,\delta u' = A(\sigma + u'\sigma)\,\delta u' = (1 + u')\,\delta u'\,\sigma A = \varepsilon_u(u, \delta u)\,\sigma A\,, \tag{8.13}$$

leads to Green's first identity

$$\mathscr{G}(u, \delta u) = \int_0^l -\mathcal{N}' \, \delta u \, dx + [\mathcal{N} \, \delta u]_0^l - \underbrace{\int_0^l \varepsilon_u(u, \delta u) \, \sigma \, A \, dx}_{a_u(u, \delta u)} = 0 \tag{8.14}$$

with $\varepsilon_u(u, \delta u) := (1 + u') \, \delta u'$ as the Gateaux derivative

$$\frac{d}{d\eta} \varepsilon(u + \eta \, \delta u)|_{\eta=0} = \frac{d}{d\eta}\big(u' + \eta \, \delta u' + \frac{1}{2} \, (u' + \eta \, \delta u')^2\big)|_{\eta=0} \tag{8.15}$$

of $\varepsilon(u)$ in the direction of δu.

8.3.1 Newton's Method

Let $u(x)$ be the longitudinal displacement of a bar fixed on the left, $u(0) = 0$, and with a free end, $N(l) = 0$. Given the solution

$$u_h = \sum_j u_j \, \varphi_j(x) \,, \tag{8.16}$$

we determine the nodal displacements u_j by solving the n equations

$$a_u(u_h, \varphi_i) - \int_0^l p \, \varphi_i \, dx = k_i(\boldsymbol{u}) - f_i = 0 \qquad i = 1, 2, \ldots, n \,. \tag{8.17}$$

This is a set $\boldsymbol{k}(\boldsymbol{u}) = \boldsymbol{f}$ of n nonlinear equations, which a computer solves iteratively with ***Newton's method***

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i - (\nabla \boldsymbol{k}(\boldsymbol{u}_i))^{-1} (\boldsymbol{k}(\boldsymbol{u}_i) - \boldsymbol{f}) \tag{8.18}$$

or

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i - \boldsymbol{K}_T^{-1}(\boldsymbol{u}_i) \, (\boldsymbol{k}(\boldsymbol{u}_i) - \boldsymbol{f}) \,, \tag{8.19}$$

where $\boldsymbol{K}_T(\boldsymbol{u}_i)$ is the ***tangential stiffness matrix***.

8.4 Geometrically Nonlinear Beam

The bending stiffness EI and longitudinal stiffness EA are constant, and the distributed loads are p_x and p_z. The displacements are the longitudinal displacement u and the deflection w, and this pair can also be written as a vector $\boldsymbol{v} = \{u, w\}^T$,

$$\varepsilon = u' + \frac{1}{2}\,(w')^2 \quad \kappa = w'' \tag{8.20a}$$

$$N = EA\,\varepsilon \quad M = -EI\,\kappa \tag{8.20b}$$

$$-N' = p_x \quad -M'' - (N\,w')' = p_z\,. \tag{8.20c}$$

The “displacement formulation” of this system is

$$-EA\,(u' + \frac{1}{2}\,(w')^2)' = p_x \tag{8.21a}$$

$$EI\,w^{IV} - (EA\,(u' + \frac{1}{2}\,(w')^2)\,w')' = p_z\,, \tag{8.21b}$$

or in a slightly more transparent formulation

$$-N' = p_x \tag{8.22a}$$

$$EI\,w^{IV} - (N\,w')' = p_z\,. \tag{8.22b}$$

Let

$$N = N(\boldsymbol{v}) = EA\,(u' + \frac{1}{2}\,(w')^2)\,, \qquad M = M(w) = -EI\,w''\,, \tag{8.23}$$

and let $\boldsymbol{L}\boldsymbol{v}$ the left side of (8.21), integration by parts of the work integral results in

$$\begin{aligned}\int_0^l \boldsymbol{L}\boldsymbol{v} \bullet \boldsymbol{\delta v}\,dx &= \int_0^l ((Eq_1)\cdot\delta u + (Eq_2)\cdot\delta w)\,dx \\ &= \int_0^l [(-N'\,\delta u - (M'' + (N\,w')')\,\delta w]\,dx \\ &= -[N\,\delta u + (M' + N\,w')\,\delta w - M\,\delta w']_0^l + a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{\delta v})\,,\end{aligned} \tag{8.24}$$

where

$$\begin{aligned}a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{\delta v}) &= \int_0^l (-M\,\delta w'' + N\,(\delta u' + w'\,\delta w'))\,dx \\ &= \int_0^l (\frac{M(w)\,M_w(w, \delta w)}{EI} + \frac{N(\boldsymbol{v})\,N_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{\delta v})}{EA})\,dx\end{aligned} \tag{8.25}$$

with the ***Gateaux derivatives*** of M and N

$$M_w(w, \delta w) = [\frac{d}{d\varepsilon} M(w + \varepsilon\,\delta w)]_{|\varepsilon=0} = \delta M \tag{8.26}$$

$$N_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{\delta v}) = [\frac{d}{d\varepsilon} N(\boldsymbol{v} + \varepsilon\,\boldsymbol{\delta v})]_{|\varepsilon=0} = EA(\delta u' + w'\delta w')\,. \tag{8.27}$$

So, Green's first identity is the expression

$$\mathscr{G}(\boldsymbol{v}, \boldsymbol{\delta v}) = \int_0^l \boldsymbol{L u} \cdot \boldsymbol{\delta u}\, dx + [N\, \delta u + (M' + N\, w')\, \delta w - M\, \delta w']_0^l$$
$$- a_v(\boldsymbol{v}, \boldsymbol{\delta v}) = 0\,. \tag{8.28}$$

If the normal force N is constant, we arrive at the well-known equation

$$EI\, w^{IV} - N\, w'' = p_z \tag{8.29}$$

of second-order theory.

8.4.1 Conservation of Energy

For a linear bar conservation of energy means

$$\frac{1}{2}\, \mathscr{G}(u, u) = \frac{1}{2} \int_0^l p\, u\, dx - \frac{1}{2} \int_0^l \frac{N^2}{EA}\, dx = W_e - W_i = 0\,. \tag{8.30}$$

In the case of the cantilever beam in Fig. 8.1, with

$$\varepsilon = l\sqrt{\frac{P_x}{EI}}\,, \qquad P_x = P_z = 1\,, \qquad l = EI = 1\,, \tag{8.31}$$

the displacements according to **second-order theory** are

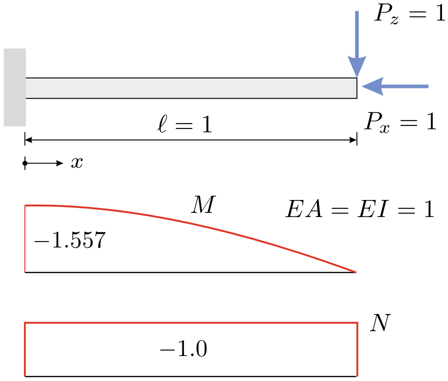

Fig. 8.1 Second-order theory

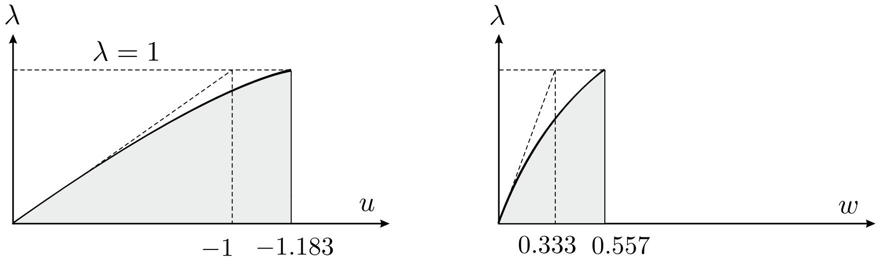

Fig. 8.2 Displacements at the end of the beam

$$u(x) = \frac{1}{\varepsilon(\sin^2(\varepsilon) - 1)} [-0.5\sin(\varepsilon(x-2)) + 3\sin(2\varepsilon) - \sin(2\varepsilon(x-1)) + 4\sin(\varepsilon x) - 6\,\varepsilon x - 8\,\varepsilon^3 x + 4\,\varepsilon x\sin^2(\varepsilon) + 8\,\varepsilon^3 x\sin^2(\varepsilon)] \tag{8.32a}$$

$$w(x) = \frac{P_z}{\varepsilon^3}[\tan(\varepsilon)\cdot(1-\cos(\varepsilon x)) + \sin(\varepsilon x) - \varepsilon x]\,. \tag{8.32b}$$

The statement $W_e = W_i$ is still true, but we cannot calculate the external work W_e as in the linear theory, see Fig. 8.2, by simply superposing the load with the displacements

$$W_e \neq \frac{1}{2}(P_x\,u(l) + P_z\,w(l)) = \frac{1}{2}(1.0\cdot 1.183 + 1.0\cdot 0.557) = 0.87\,, \tag{8.33}$$

but instead we must count the increments. Let $\boldsymbol{dv} = \{du, dw\}$ be the displacement increment with respect to a load increment, then we read

$$dW_i = a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{dv}) = \int_0^l \Big(\frac{M(w)\,M_w(w,dw)}{EI} + \frac{N(\boldsymbol{v})\,N_{\boldsymbol{v}}(\boldsymbol{v},\boldsymbol{dv})}{EA}\Big)\,dx \tag{8.34}$$

as the increase in internal energy and Conservation of energy

$$\int dW_i = \int_0^l \Big[\int_0^M \frac{\bar{M}\,d\bar{M}}{EI} + \int_0^N \frac{\bar{N}\,d\bar{N}}{EA}\Big]\,dx = \frac{1}{2}\int_0^l \Big(\frac{N^2}{EA} + \frac{M^2}{EI}\Big)\,dx = 0.9670 \tag{8.35}$$

is therefore the internal energy at the end of the load path.

In order to calculate the external work, we assume the load to slowly increase,

$$P_x(\lambda) = \lambda\,P_x = \lambda\cdot 1 \qquad P_z(\lambda) = \lambda\,P_z = \lambda\cdot 1 \qquad 0 \leq \lambda \leq 1\,. \tag{8.36}$$

so that the external work is [2], p. 333,

$$\int dW_e = \int_0^u \lambda \cdot 1\, d\bar{u} + \int_0^w \lambda \cdot 1\, d\bar{w} = \int_0^1 (\bar{u},_\lambda + \bar{w},_\lambda)\, d\lambda\,. \tag{8.37}$$

This integral is best evaluated numerically, $\Delta = 0.05$,

$$\int dW_e = \int_0^1 [\frac{\bar{u}(\lambda+\Delta) - \bar{u}(\lambda)}{\Delta} + \frac{\bar{w}(\lambda+\Delta) - \bar{w}(\lambda)}{\Delta}] d\lambda \tag{8.38}$$

and with *Simpson's rule* we obtain

$$\int dW_e = 0.9669\,, \tag{8.39}$$

which matches the internal energy quite well.[1]

8.5 Geometrically Nonlinear Kirchhoff Plate

The formulation is the same as for the geometrically nonlinear beam, it only consists of a more extensive set of equations, and so we refer interested readers to pp. 325–328 in [2].

8.6 Nonlinear Elasticity Theory

In the triplet $\{\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{S}\}$ the tensor $\boldsymbol{E}$ is the Green-Lagrange strain tensor and $\boldsymbol{S}$ is the second Piola-Kirchhoff stress tensor. We assume the material to be hyperelastic, i.e. there exists a strain energy function $\boldsymbol{W}$ such that $\boldsymbol{S} = \partial \boldsymbol{W}/\partial \boldsymbol{E}$.

In the presence of volume loads $\boldsymbol{p}$ the elastic state $\boldsymbol{\Sigma} = \{\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{S}\}$ at each point $\boldsymbol{x}$ of the undeformed body satisfies the system[2]

$$\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E} = \boldsymbol{0} \qquad \frac{1}{2}(u_i,_j + u_j,_i + u_k,_i\, u_k,_j) - \varepsilon_{ij} = 0 \tag{8.40a}$$

$$\boldsymbol{W}'(\boldsymbol{E}) - \boldsymbol{S} = \boldsymbol{0} \qquad \frac{\partial W}{\partial \varepsilon_{ij}} - \sigma_{ij} = 0 \tag{8.40b}$$

$$-\text{div}(\boldsymbol{S} + \nabla \boldsymbol{u} \boldsymbol{S}) = \boldsymbol{p} \qquad -(\sigma_{ij} + u_i,_k\, \sigma_{kj}),_j = p_i \tag{8.40c}$$

[1] Since $EA = 1$ the example is of course academic because the compression exceeds the length of the bar.

[2] In linear 1-D problems $\sigma = E\,\varepsilon$, $\boldsymbol{W}(\boldsymbol{E}) = 0.5\, E\, \varepsilon^2$.