Chapter 5
Stiffness Changes and Reanalysis

The subject of this chapter are changes in the stiffness of single structural members and we study how such shifts propagate through a structure, how they affect the stress distribution in a structure.

In terms of linear algebra such modifications correspond to an update of the stiffness matrix, $\boldsymbol{K} \rightarrow \boldsymbol{K} + \boldsymbol{\Delta K}$, and the solution $\boldsymbol{u} \rightarrow \boldsymbol{u}_c = \boldsymbol{u} + \boldsymbol{\Delta u}$

$$(\boldsymbol{K} + \boldsymbol{\Delta K})\,(\boldsymbol{u} + \boldsymbol{\Delta u}) = \boldsymbol{f} \qquad \text{or} \qquad \boldsymbol{K}_c\,\boldsymbol{u}_c = \boldsymbol{f}\,. \tag{5.1}$$

The approach in this chapter is that we compute the response $\boldsymbol{u}_c$ of the modified structure via the response $\boldsymbol{u}$ of the original, unmodified structure $\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}$, a technique which is called *reanalysis*.

The essential insight is: The new displacement vector $\boldsymbol{u}_c$ is the response of the original system when a vector $\boldsymbol{f}^+ = -\boldsymbol{\Delta K}\,\boldsymbol{u}_c$ is added to the right side

$$\boldsymbol{K}\boldsymbol{u}_c = \boldsymbol{f} + \boldsymbol{f}^+\,, \tag{5.2}$$

and $\boldsymbol{f}^+$ is orthogonal to all rigid-body motions $\boldsymbol{u}_0 = \boldsymbol{a} + \boldsymbol{x} \times \boldsymbol{b}$, which is the reason why the effects of local stiffness changes most often subside rapidly.

A stiffness change triggers a compensating displacement $\boldsymbol{\Delta u} = \boldsymbol{u}_c - \boldsymbol{u}$, and $\boldsymbol{\Delta u} = \boldsymbol{K}^{-1}\boldsymbol{f}^+$ is the reaction of the structure to additional self-equilibrated nodal forces $\boldsymbol{f}^+ = -\boldsymbol{\Delta K}\boldsymbol{u}_c$, that is $\boldsymbol{f}^+ \bullet (\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}) = 0$

The problem with this approach is of course that the new vector $\boldsymbol{u}_c$, on which $\boldsymbol{f}^+ = \boldsymbol{\Delta K}\,\boldsymbol{u}_c$ depends, is unknown, though eventually the vector $\boldsymbol{\Delta K u} \simeq \boldsymbol{f}^+$ can serve as substitute. We will also present two techniques to calculate $\boldsymbol{u}_c$ directly, see Sect. 5.22, but our main concern is not to beat the computer, but we want insight into the effects of stiffness changes.

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The algebra behind reanalysis is easily explained: To solve the equation

$$-(H + \Delta H)u_c'' = p \tag{5.3}$$

we place the term $-\Delta H\, u_c''$ on the right side,

$$-H\, u_c'' = p + \Delta H\, u_c'' \tag{5.4}$$

and so the equivalent nodal forces are

$$\int_0^l \varphi_i\, p\, dx + \int_0^l \varphi_i\, \Delta H\, u_c''\, dx = f_i + f_i^+ \,. \tag{5.5}$$

Back to $\boldsymbol{K}\,\boldsymbol{u}_c = \boldsymbol{f} + \boldsymbol{f}^+$. The new solution $\boldsymbol{u}_c$ is, like the old solution $\boldsymbol{u} = \sum_i f_i\, \boldsymbol{g}_i$, an expansion in terms of the influence functions of the nodal displacements (= the columns $\boldsymbol{g}_i$ of the **old** inverse $\boldsymbol{K}^{-1}$)

$$\boldsymbol{u}_c = \sum_i f_i\, \boldsymbol{g}_i + f_a^+\, \boldsymbol{g}_a + f_b^+\, \boldsymbol{g}_b + \cdots = \boldsymbol{u} + f_a^+\, \boldsymbol{g}_a + f_b^+\, \boldsymbol{g}_b + \cdots \,, \tag{5.6}$$

only that some columns appear twice, and then carry **additional weights** $f_a^+, f_b^+, \ldots$ corresponding to the modified values k_{ab}. If for example four entries $k_{3,3}$, $k_{3,7}$, $k_{7,3}$, $k_{7,7}$ change, two (initially unknown) weights f_3^+ and f_7^+ mark the difference between the new and the old displacement vector

$$\boldsymbol{u}_c - \boldsymbol{u} = f_3^+\, \boldsymbol{g}_3 + f_7^+\, \boldsymbol{g}_7 \,. \tag{5.7}$$

Nothing new is added, the old is only supplemented—with old.

This is the same approach as in the **force method**, since we do not change the stiffness matrix, but we change the right side, $\boldsymbol{f}$ becomes $\boldsymbol{f} + \boldsymbol{f}^+$.

The force method chooses a statically determinate structure as its *primary structure*, and in the sequence all calculations are done on this system. It corresponds to the undisturbed system $\boldsymbol{K}$ since the redundants X_i play the same role as the f_i^+. While the f_i^+ couple the added element to the structure, the X_i make that the gaps at the joints close. Both, the X_i and f_i^+, are *additional loads* which appear on the right side, while the proper analysis is done with the primary structure (matrix $\boldsymbol{K}$).

This approach has a further advantage: We do not need two sets of influence functions, one for the primary structure (matrix $\boldsymbol{K}$) and a separate set for the statically indeterminate structure (matrix $\boldsymbol{K}_c$), since the changes in any functional

$$J(\boldsymbol{e}) = J(\boldsymbol{u}_c) - J(\boldsymbol{u}) = \boldsymbol{g}^T \boldsymbol{f}^+ \,, \tag{5.8}$$

can be calculated with the influence functions $\boldsymbol{g}$ of the original system $\boldsymbol{K}$.

5.1 Parameter Identification

If you pull on a spring with a force f and measure the extension u, you can determine the stiffness $k = f/u$ of the spring. Parameter identification means fitting the elements k_{ij} of a stiffness matrix to measured data. Mathematically this counts as a (difficult) **inverse problem**. Since the columns $\boldsymbol{g}_i$ of $\boldsymbol{K}^{-1}$ are the influence functions of the nodal displacements, everything depends on the influence functions. How do the system responses change with corrections $k_{ij} \to k_{ij} + \Delta k_{ij}$? How to modify the k_{ij} to reproduce the measured data? How sensitive is the structure to such changes?

These are precisely the questions that also play an important role in reanalysis, because in Sect. 5.14 we will see that the derivative of the displacement vector with respect to an element k_{ij}

$$\frac{\partial \boldsymbol{u}}{\partial k_{ij}} = -\boldsymbol{K}^{-1} u_j \, \boldsymbol{e}_i = -u_j \, \boldsymbol{g}_i \,, \tag{5.9}$$

depends on the influence function $\boldsymbol{g}_i$ and the nodal displacement u_j. In this sense the title of the chapter could also have been *Reanalysis and Parameter Identification*. We could extend this list also further by adding *Structural Health Monitoring* which makes ample use of parameter identification techniques, but then mainly in the dynamic range [1].

5.2 Introductory Remarks

We start with a spring, $k\,u = f$, see Fig. 5.1. The reaction of the spring to a force $f = 1$ (the Dirac delta) is $G = 1/k$, and a shift in the stiffness, $k + \Delta\, k$, changes the response to $G_c = 1/(k + \Delta\, k)$, and so the reaction to any force f before, u, and after, u_c, is

$$u = \frac{1}{k} f \qquad u_c = \frac{1}{k + \Delta\, k} f \,. \tag{5.10}$$

A **Taylor series** of the updated influence function

$$\frac{1}{k + \Delta\, k} = \frac{1}{k} - \frac{1}{k^2}\, \Delta\, k + \cdots \tag{5.11}$$

illustrates, how the spring reacts to the shift in the stiffness

$$u_c \simeq \left[\frac{1}{k} - \frac{1}{k}\, \frac{\Delta\, k}{k}\right] f = u - \frac{1}{k}\, \underbrace{\Delta\, k \cdot u}_{force} \,. \tag{5.12}$$

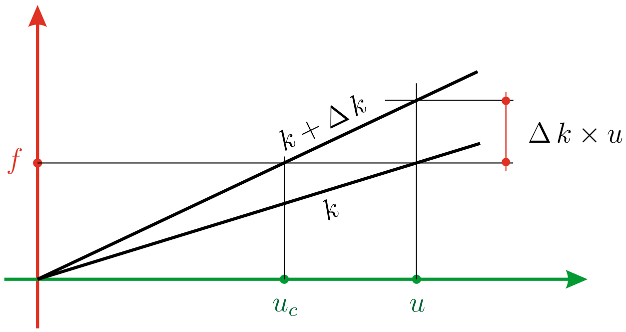


Fig. 5.1 A change in the stiffness means a change in the slope of $k\,u = f$

The increase, $k \to k + \Delta\,k$, lets the original u shoot beyond the target, the force in the spring, $(k + \Delta\,k)\,u = f + \Delta\,k \cdot u$, becomes too large, and to eliminate this too much an opposite displacement $\Delta\,u \simeq -\Delta\,k \cdot u/k$ must correct this.

Since the Taylor series of a stiffness matrix reads [2],

$$(\boldsymbol{K} + \boldsymbol{\Delta\,K})^{-1} = \boldsymbol{K}^{-1} - \boldsymbol{K}^{-1}\boldsymbol{\Delta\,K}\,\boldsymbol{K}^{-1} + \cdots \tag{5.13}$$

the analogy between (5.12) and

$$\boldsymbol{u}_c = (\boldsymbol{K} + \boldsymbol{\Delta\,K})^{-1}\boldsymbol{f} \simeq \boldsymbol{u} - \boldsymbol{K}^{-1}\boldsymbol{\Delta\,K}\,\boldsymbol{u} \tag{5.14}$$

is evident. If we multiply this equation from the left with $\boldsymbol{K}$

$$\boldsymbol{K}\boldsymbol{u}_c = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{\Delta K}\boldsymbol{u}\,, \tag{5.15}$$

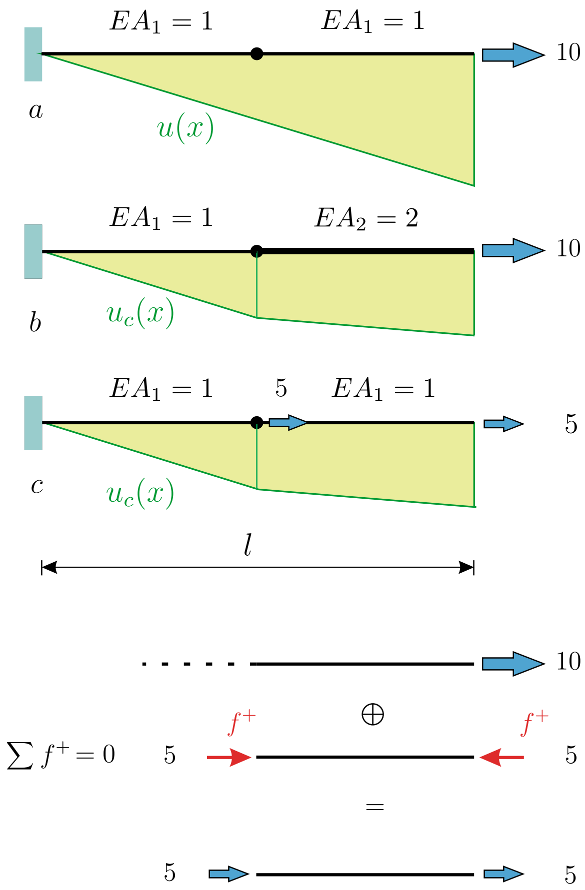
it has (almost) the form in which we treat it in this chapter. Almost, since we do not use a Taylor series, but the exact formula, i.e. we replace $-\boldsymbol{\Delta K}\boldsymbol{u}$ with the exact $-\boldsymbol{\Delta K}\boldsymbol{u}_c =: \boldsymbol{f}^+$

$$\boldsymbol{K}\boldsymbol{u}_c = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{\Delta K}\boldsymbol{u}_c = \boldsymbol{f} + \boldsymbol{f}^+\,. \tag{5.16}$$

The approximation (5.14) means $\boldsymbol{u}_c - \boldsymbol{u} \simeq -\boldsymbol{K}^{-1}\boldsymbol{\Delta K}\boldsymbol{u}$. We can even understand this equation, if we apply it to a beam, $EI \to EI + \Delta EI$, and write it in integral form

$$w_c(x) - w(x) = -\int_0^l \underbrace{G_0(y,x)}_{\boldsymbol{K}^{-1}}\;\underbrace{\Delta\,EI\,\frac{d^4}{dy^4}}_{\boldsymbol{\Delta\,K}}\;\underbrace{\int_0^l G_0(y,z)\,p(z)\,dz}_{\boldsymbol{u}}\;dy \tag{5.17}$$

Fig. 5.2 In the second element the stiffness *EA* is doubled



or

$$w_c(x) - w(x) = -\int_0^l G_0(y,x)\,\frac{\Delta\,EI}{EI}\,p(y)\,dy\,. \tag{5.18}$$

So, a decrease, $\Delta EI < 0$, is equivalent to an increase in the load p.[1]

Example 5.1 An elementary example may illustrate the technique. The bar in Fig. 5.2 consists of two elements with the same stiffness $EA = 1$ kN, and a force $f_2 = 10$ kN pulls at its right end

[1]It is $EI\,d^4/dy^4 G_0 = \delta_0$ and if you replace EI by ΔEI, this is $\Delta EI/EI\,\delta_0$ and so the integral gives $(\Delta EI/EI\,\Delta_0, p) = \Delta EI/EI \cdot p(y)$. Furthermore, $\boldsymbol{\Delta K u} = \delta W_i = \delta W_e$ and δW_e contains the operator $\Delta EI\,d^4/dy^4$.

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \tag{5.19}$$

and so, $u_1 = 10\,\text{m}$, $u_2 = 20$ m.

On doubling the stiffness of the second element, $EA \to 2EA$,

$$\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_1^c \\ u_2^c \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \tag{5.20}$$

the nodal displacements become $u_1^c = 10\,\text{m}$, $u_2^c = 15$ m.

The question is then which vector $\boldsymbol{f}^+$ we must add to the right side $\boldsymbol{f}$ of the original system (5.19) to produce the same effect, the same displacements

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^c \\ u_2^c \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + X \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{5.21}$$

We assume the vector $\boldsymbol{f}^+$ to be an equilibrium vector (the sum of its components is zero), and the result justifies this assumption since with $X = 5$ we obtain a solution, a vector

$$\boldsymbol{f} + \boldsymbol{f}^+ = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \tag{5.22}$$

to which the original system reacts in the same way as the modified system, the nodal displacements $u_i = u_i^c$ are the same

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}. \tag{5.23}$$

5.3 Adding or Subtracting Stiffness

A change in the stiffness of an element can be interpreted as placing an **additional element** in front of the original element, so that the two elements together have the target stiffness, see Fig. 5.3.

Since the additional element must be synchronized with the motions of the original structure **coupling forces** need to hold the nodes together, and these forces are just the $\boldsymbol{f}^+$. This explains why the forces $\boldsymbol{f}^+$ are equilibrium forces, because if they were not, the added element would fly off.

If the stiffness of the element increases, the coupling forces $\boldsymbol{f}^+$ normally have the tendency to hinder the deformation of the element, they stiffen, so to speak, the element.

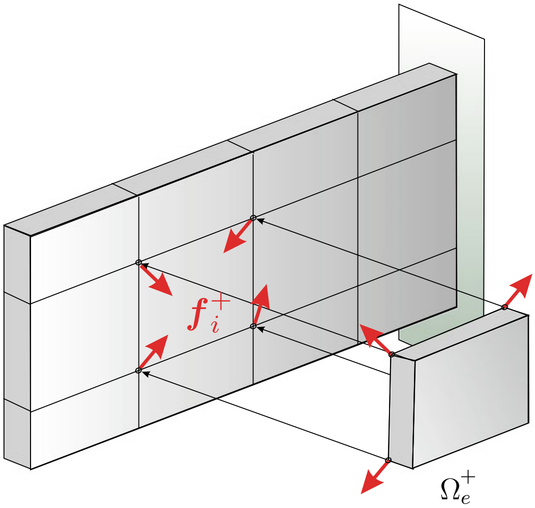


Fig. 5.3 A stiffness change $\boldsymbol{K} + \boldsymbol{\Delta K}$ is equivalent to attaching an element Ω_e^+ having the stiffness matrix $\boldsymbol{\Delta K}$ to the structure [3]

Conversely, if the stiffness in the element decreases, the coupling forces $\boldsymbol{f}^+$ will add to the deformations of the element, they act like additional weights at the nodes of the element.

5.4 Dipoles and Monopoles

Two opposite forces $f_i^+ = \pm 1/h$, a distance h apart, become a dipole when h tends to zero.

If, however, the two forces remain finite even in the limit $h = 0$, we call this a ***pseudodipole***. The proton (+) and the electron (−) in a hydrogen atom form such a pseudodipole, since the distance between the two opposite elementary charges is so small that their effects on a point charge outside the atom practically cancel out.

The same can be said about the forces f_i^+, since to each force f_i^+ corresponds an opposite force f_j^+, so that the two forces f^+, seen from a distance, resemble a pseudodipole, see Fig. 5.4.

The effect of the forces f_i^+ on any point x of the structure depends on how large the **runtime difference** from the point x to the force $+f_i^+$ and the counterforce $-f_i^+$ is. If two forces $\pm f_i^+$ are only a small distance apart, because the element Ω_e is small, their effects cancel each other out, since the influence function hardly changes on the element, $G' \simeq 0$.

Imagine a bar, which is modeled with seven linear elements, and in the second element the stiffness changes, $EA_c = EA + \Delta EA$, while we are about to calculate the influence function for the normal force in the last element by spreading the element at its midpoint by one unit.

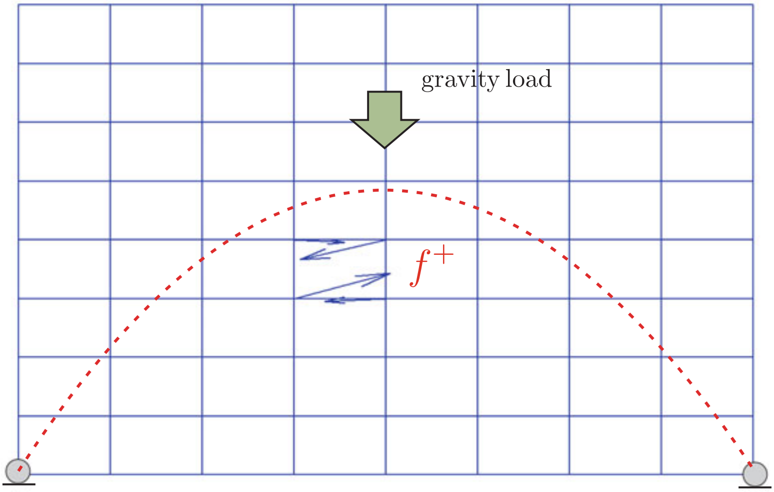


Fig. 5.4 A stiffness change in an element activates coupling forces f_i^+. These forces follow the directions of the principal stresses (- - -) and they are equilibrium forces, comparable to pseudodipoles

The effect of the spread propagates to the left, to element # 2, where two forces $\pm f^+$ (at the two nodes of the element) simulate the effect of the stiffness change, $EA_c = EA + \Delta EA$. One force f_i^+ pulls to the left and an opposite force f_{i+1}^+ pulls to the right. The influence function for the normal force

$$G(y, x) = \sum_j g_j(x)\,\varphi_j(y) \tag{5.24}$$

will have the value g_i at the left node and g_{i+1} at the right node of element # 2, and so the effect of the stiffness change on the normal force $N = J(u)$, *new–old*, in the last element

$$N_c - N = J(u_c) - J(u) = f_i^+ g_i - f_{i+1}^+ g_{i+1} = f_i^+ \cdot (g_i - g_{i+1}) \simeq f_i^+ \cdot G' \cdot l_e\,, \tag{5.25}$$

will only be noticeable if the influence function has a distinct slope, $G' \gg 0$, in the element #2. We may express this observation by saying:

The force pair $\pm f_i^+$ "differentiates" the influence function.

Remark 5.1 This picture of two forces f_i^+ and f_{i+1}^+ exactly opposite each other and of the same size, is only true in the 1-D model of a bar. In a beam element the f_i^+ would be two forces (which still add to zero) and two moments and all four balance