Chapter 4
Betti Extended

In the previous chapter we repeatedly made use of the fact that the FE-solution $u_h(x)$ is the superposition of the **approximate influence function** $G_h(y, x)$ and the load $p(y)$

$$u_h(x) = \int_0^l G_h(y, x)\, p(y)\, dy\,. \tag{4.1}$$

This result is based on a theorem which we call *Betti extended* .

Theorem 4.1 (*Betti extended*) *One may replace the exact solutions u_1 and u_2 in Betti's theorem*

$$W_{1,2} = \int_0^l p_1 \underset{\uparrow}{u_2}\, dx = \int_0^l p_2 \underset{\uparrow}{u_1}\, dx = W_{2,1} \tag{4.2}$$

with the FE-approximations u_{1h} and u_{2h}

$$W^h_{1,2} = \int_0^l p_1 \underset{\uparrow}{u_{2h}}\, dx = \int_0^l p_2 \underset{\uparrow}{u_{1h}}\, dx = W^h_{2,1}\,. \tag{4.3}$$

The claim is not that $W^h_{1,2}$ is the same as $W_{1,2}$, but rather: If $W_{1,2} = W_{2,1}$ is true, then $W^h_{1,2} = W^h_{2,1}$ is true as well; in short,

$$(p_1, u_2) = (p_2, u_1) \qquad \Rightarrow \qquad (p_1, u_{2h}) = (p_2, u_{1h})\,. \tag{4.4}$$

The best way to understand *Betti extended* is to focus on ***Maxwell's theorem***, see Fig. 4.1, which is just a particular application of *Betti's theorem*

$$P_1 \cdot w_2(x_1) = P_2 \cdot w_1(x_2)\,. \tag{4.5}$$

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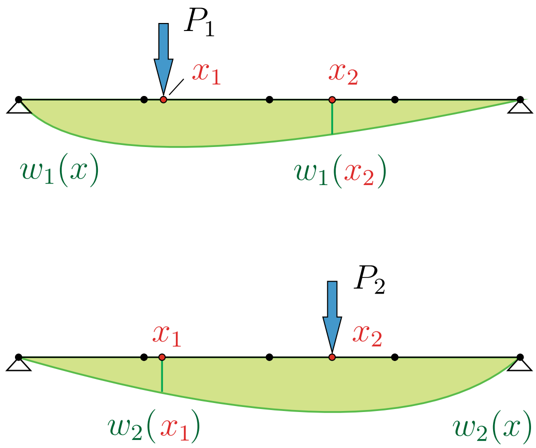
Fig. 4.1 *Maxwell's theorem*



If the two curves $w_1(x)$ and $w_2(x)$ are approximated with finite elements, the deflections at the two points x_1 and x_2 are not exact

$$w_{1h}(x_2) \neq w_1(x_2) \qquad w_{2h}(x_1) \neq w_2(x_1)\,, \tag{4.6}$$

but *Betti extended* guarantees that *Maxwell's theorem* holds true also in this situation

$$P_1 \cdot w_{2h}(x_1) = P_2 \cdot w_{1h}(x_2)\,. \tag{4.7}$$

This feature extends *Maxwell's theorem* to FE-solutions, set $P_1 = P_2 = 1$, which is not necessarily self-evident. That it had to be true for the nodal values, is a consequence of the symmetry of the stiffness matrices. ***Betti extended*** guarantees this also for all points in between.

4.1 Proof

The proof of ***Betti extended*** is based on the two equations

$$\int_\Omega p_{1h}\, u_{2h}\, d\Omega = \int_\Omega p_1\, u_{2h}\, d\Omega \tag{4.8a}$$

$$\int_\Omega p_{2h}\, u_{1h}\, d\Omega = \int_\Omega p_2\, u_{1h}\, d\Omega\,, \tag{4.8b}$$

and ***Betti's theorem*** itself

$$\mathscr{B}(u_{1h}, u_{2h}) = \int_\Omega p_{1h}\, u_{2h}\, d\Omega - \int_\Omega p_{2h}\, u_{1h}\, d\Omega = 0\,. \tag{4.9}$$

This gives

$$\int_{\Omega} p_1\, u_{2h}\, d\Omega = \int_{\Omega} p_{1h}\, u_{2h}\, d\Omega = \int_{\Omega} p_{2h}\, u_{1h}\, d\Omega = \int_{\Omega} p_2\, u_{1h}\, d\Omega\,, \tag{4.10}$$

or

$$\int_{\Omega} p_1\, u_{2h}\, d\Omega = \int_{\Omega} p_2\, u_{1h}\, d\Omega\,, \tag{4.11}$$

which is *Betti extended*.

At Eq. (4.8a) we arrive as follows: According to the *Galerkin-orthogonality* we have

$$\delta W_i = a(u_1 - u_{1h}, \varphi_i) = 0\,, \tag{4.12}$$

or, if we write it as external instead of internal virtual work, $\delta W_i = \delta W_e$,

$$\int_{\Omega} (p_1 - p_{1h})\, \varphi_i\, d\Omega = 0 \quad i = 1, 2, \ldots n \quad \Rightarrow \quad \int_{\Omega} (p_1 - p_{1h})\, u_{2h}\, d\Omega = 0\,, \tag{4.13}$$

since u_{2h} is a linear combination of the φ_i. In the same way we arrive at the second equation.

With *Betti extended* the proof of the central Eq. (4.1) is easy, since in the influence function for $u(x)$

$$W_{1,2} = 1 \cdot u(x) = \int_0^l \delta(y - x)\, u(y)\, dy = \int_0^l G(y, x)\, p(y)\, dy = W_{2,1}\,, \tag{4.14}$$

we may replace u and G with the FE-solutions u_h and G_h

$$W^h_{1,2} = \int_0^l \delta(y - x)\, \underset{\uparrow}{u_h}(y)\, dy = \int_0^l \underset{\uparrow}{G_h}(y, x)\, p(y)\, dy = W^h_{2,1}\,, \tag{4.15}$$

and so

$$u_h(x) = \int_0^l G_h(y, x)\, p(y)\, dy\,. \tag{4.16}$$

This **switch**, $u \rightarrow u_h$ and $G \rightarrow G_h$, can be applied to all linear functionals

$$J(u) = \int_0^l \delta(y - y)\, u(y)\, dy = \int_0^l G(y, x)\, p(y)\, dy\,, \tag{4.17}$$

resulting in

$$J(u_h) = \int_0^l \delta(y - y)\, u_h(y)\, dy = \int_0^l G_h(y, x)\, p(y)\, dy\,. \tag{4.18}$$

4.2 At Which Points Is the FE-Solution Exact?

With the help of *Betti extended* we can now also specify when and where FE-results are exact.

We study this question with a prestressed rope. The influence function $G(y, x)$ for the deflection $u(x)$ of the rope, see Fig. 4.2, at the point $x = 1.5$ is the response of the rope to a single force $P = 1$, a **Dirac delta** $\delta(y - x)$.

Since the FE-program cannot generate this shape, it instead places two half as large single forces at the two neighboring nodes. This is—in our notation—the load case $\delta_h(y, x)$ and the corresponding deflection $G_h(y, x)$ is the approximate influence function.

So, there are **two Dirac deltas**, the exact and the approximate

$$\delta(y - x) \quad \downarrow \qquad \delta_h(y - x) \quad \frac{1}{2} \downarrow + \frac{1}{2} \downarrow\,, \tag{4.19}$$

and also, **two influence functions**

$$G(y, x) \quad \text{(one peak)} \qquad G_h(y, x) \quad \text{(two peaks)}\,. \tag{4.20}$$

With finite elements we search for an approximate solution in the space $\mathcal{V}_h$, i.e. all the rope polygons, which can be generated with the three nodal shape functions $\varphi_i(x)$. Note, the dual of $\mathcal{V}_h$ is the space $\mathcal{V}_h^*$ of all load cases (nodal forces f_1, f_2, f_3), which create the rope polygons in $\mathcal{V}_h$.

Now, if a function u_h lies in $\mathcal{V}_h$ (is a rope polygon), the approximate Dirac delta (2 half-sized single forces) is as good as the exact Dirac delta (one single force)

$$u_h(x) = \int_0^l \delta(y - x)\, u_h(y)\, dy = \int_0^l \delta_h(y - x)\, u_h(y)\, dy\,. \tag{4.21}$$

In concrete terms this means

$$1 \cdot u_h(x) = \frac{1}{2} \cdot u_h(x_1) + \frac{1}{2} \cdot u_h(x_2)\,, \tag{4.22}$$

and this makes sense since the height of a straight line in between two nodes is just the average of the nodal values.

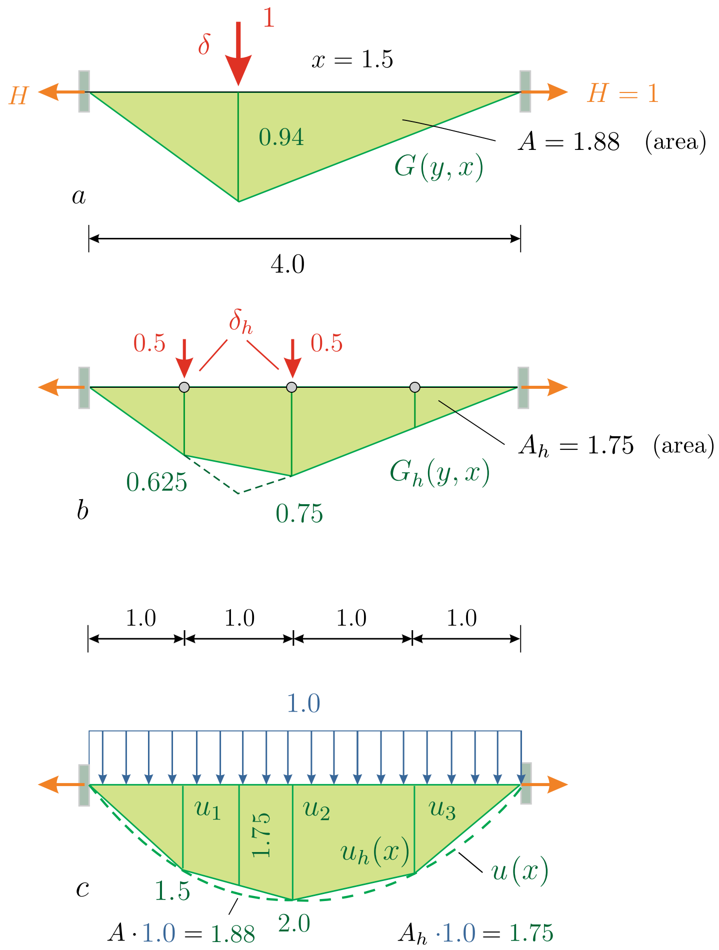


Fig. 4.2 Influence function for the deflection at the point $x = 1.5$, **a** exact influence function, **b** approximation, **c** FE-solution under uniform load $p = 1$

And because the **FE-load case** p_h lies in $\mathcal{V}_h^*$, i.e. consists of three nodal forces, the approximate influence function $G_h(y, x)$ is as good as the exact influence function

$$u_h(x) = \int_0^l G(y, x)\, p_h(y)\, dy = \int_0^l G_h(y, x)\, p_h(y)\, dy\,. \tag{4.23}$$

This too, is easy to understand. Since the load p_h consists of nodal forces f_i, the influence function is a sum over the nodes

$$u_h(x) = \int_0^l G_h(y,x)\, p_h(y)\, dy = \sum_{i=1}^{3} G_h(y_i,x)\, f_i\,. \tag{4.24}$$

But each of the three nodal influence functions is exact, $G_h(y_i,x) = G(y_i,x)$, and this explains why the sum in (4.24) is $u_h(x)$ at each point x.

On $\mathcal{V}_h$ and $\mathcal{V}_h^*$ the results obtained with $\delta_h(y,x)$ and $G_h(y,x)$ respectively are exact.

And this is true for **any** function in $\mathcal{V}_h$, since (4.21) applies to all $u_h \in \mathcal{V}_h$. And since p_h lies in $\mathcal{V}_h^*$ one can calculate any value $u_h(x)$ of the rope polygon with the approximate influence function $G_h(y,x)$. This is the essence of (4.24).

But that is not the end of it. The approximate Dirac delta, the two "half-sized" point loads at the neighboring nodes, constitute themselves a functional

$$J_h(u) = \int_0^l \delta_h(y-x)\, u(y)\, dy = \frac{1}{2}\,(u(x_1) + u(x_2))\,, \tag{4.25}$$

which can be applied to any function—not just the rope polygons in $\mathcal{V}_h$. Applied to $u(x) = \sin \pi\, x/4$ the result is

$$J_h(u) = \frac{1}{2}\,(\sin \frac{1.0\,\pi}{4} + \sin \frac{2.0\,\pi}{4}) = 0.85\,, \tag{4.26}$$

while $J(u) = \sin(1.5\,\pi/4) = 0.92$. So, there is a difference between J and J_h.

However, regarding the exact solution $u(x)$ and its FE-approximation $u_h(x)$, the following ***h-permutation rule*** applies

$$\boxed{J_h(u) = J(u_h)} \tag{4.27}$$

which can easily be verified, since

$$J_h(u) = \frac{1}{2}\,(u(1.0) + u(2.0)) = \frac{1}{2}\,(1.5 + 2.0) = 1.75 \tag{4.28}$$

$$J(u_h) = u_h(1.5) = 1.75\,. \tag{4.29}$$

The functional $J_h(u)$ measures u at the two points $x = 1.0$ and $x = 2.0$, while the functional $J(u_h)$ measures u_h only at the source point $x = 1.5$. But both measurements produce the same result!

The *h-permutation rule* is based on the fact that an FE-solution can be written in six separate ways

$$
\begin{aligned}
u_h(x) &= \int_0^l G(y,x)\, p_h(y)\, dy = \int_0^l G_h(y,x)\, p(y)\, dy \\
&= \int_0^l G_h(y,x)\, p_h(y)\, dy \\
&= \int_0^l \delta(y,x)\, u_h(y)\, dy = \int_0^l \delta_h(y,x)\, u_h(y)\, dy \\
&= \int_0^l \delta_h(y,x)\, u(y)\, dy\,, \qquad (4.30)
\end{aligned}
$$

and if we also count the three formulas

$$
u_h(x) = a(G, u_h) = a(G_h, u_h) = a(G_h, u)\,, \qquad (4.31)
$$

which are **variants of Mohr's integral**, then there are even nine.

The first two equations

$$
J(u_h) = \int_0^l G(y,x)\, p_h(y)\, dy = \int_0^l G_h(y,x)\, p(y)\, dy = J_h(u) \qquad (4.32)
$$

formulate the *h-permutation rule*, which is of course applicable to any functional $J(u_h)$, not only the displacements $u(x)$.

Whether we superpose the exact influence function G with the FE-load p_h, or the approximate influence function G_h with the original load p, makes no difference—the result is the same.

Figure 4.3 illustrates this with a pier placed under a plate, which carries a heavy-duty truck. The exact influence for the pier reaction is plotted in Fig. 4.3a and the approximate function in Fig. 4.3b. The wheel loads of the truck represent the load p, and the block load is a graphical substitute for the FE-load p_h. We have only one formula for the exact pier reaction

$$
R = \int_\Omega G(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}}\,, \qquad (4.33)
$$

but **three ways** to calculate the approximate pier reaction R_h

$$
R_h = \int_\Omega G(\boldsymbol{y}, \boldsymbol{x})\, p_h(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} = \int_\Omega G_h(\boldsymbol{y}, \boldsymbol{x})\, p_h(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} = \int_\Omega G_h(\boldsymbol{y}, \boldsymbol{x})\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}}\,. \qquad (4.34)
$$

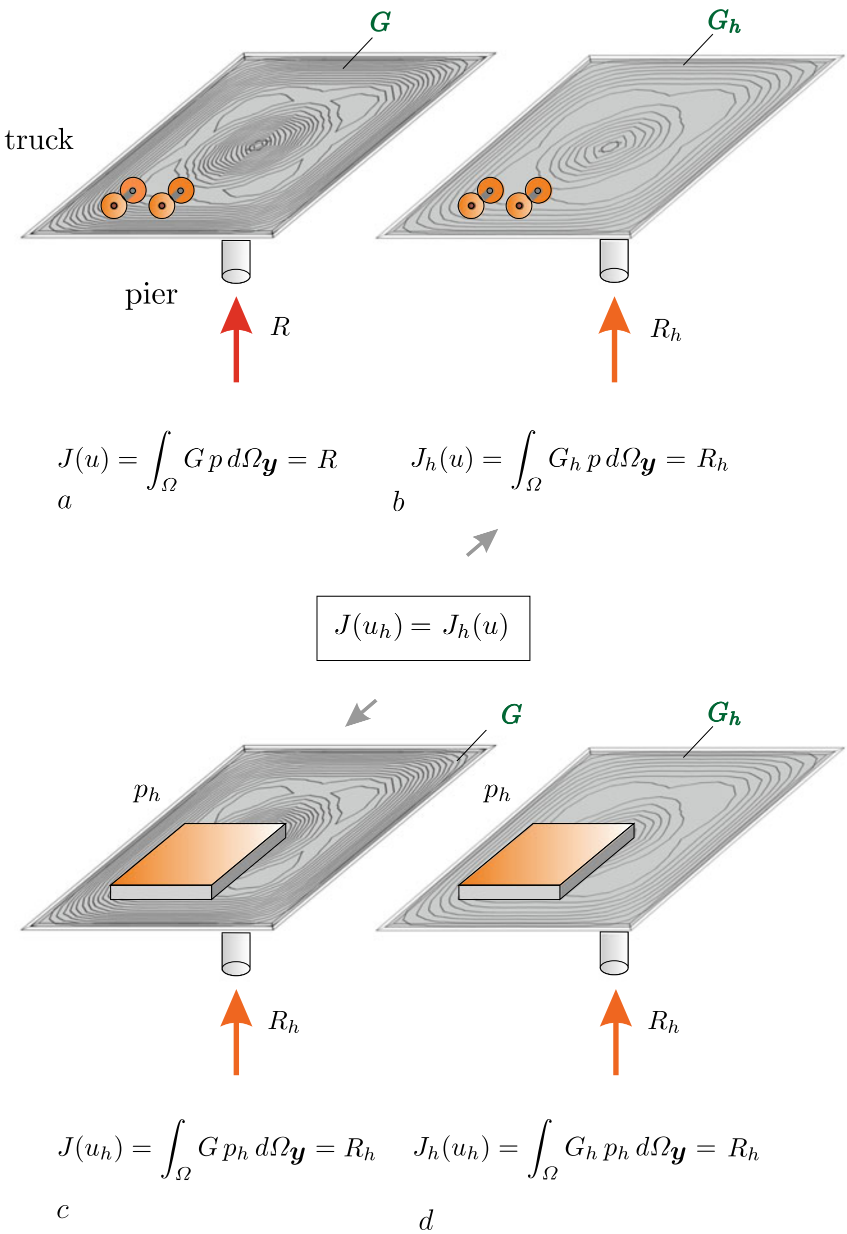


Fig. 4.3 Hinged plate with central pier. The four wheels of the truck, LC p, placed on the influence surface G_h gives the pier reaction R_h of the FE-solution. The same result is obtained, if the FE-load p_h (here pictured as a block load) is placed on the exact influence surface. And it is as well $R_h = (G_h, p) = (G_h, p_h)$, see Figure b and d

4.3 Exact Values

We can now also state, when the FE-solution is exact at a point.

Theorem 4.2 (***Exact values***)

Sufficient conditions

1. *If the influence function G of a functional J lies in $\mathcal{V}_h$, its FE-approximation G_h is identical with G and therefore*

$$J(u_h) = J_h(u) = J(u)\,, \tag{4.35}$$

or

$$J(u_h) = (G, p_h) = (G_h, p) = (G, p) = J(u)\,. \tag{4.36}$$

2. *If the exact solution lies in $\mathcal{V}_h$, $u = u_h$, the error in any influence function is orthogonal to the right side p of the solution*

$$J(u) - J(u_h) = \int_\Omega (G(\boldsymbol{y}, \boldsymbol{x})) - G_h(\boldsymbol{y}, \boldsymbol{x}))\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} = 0\,. \tag{4.37}$$

Necessary condition

1. *If a value is exact, $J(u_h) = J(u)$, the error in the influence function must be orthogonal to the right side p*

$$J(u) - J(u_h) = \int_\Omega (G(\boldsymbol{y}, \boldsymbol{x})) - G_h(\boldsymbol{y}, \boldsymbol{x}))\, p(\boldsymbol{y})\, d\Omega_{\boldsymbol{y}} = 0\,. \tag{4.38}$$

4.4 One-Dimensional Problems

Since all influence functions are piecewise homogeneous solutions of the governing differential equation, exact nodal values, $u_h(x_i) = u(x_i)$, require the trial space $\mathcal{V}_h$ to contain these solutions.

In 1-D problems such as $-EA\,u'' = p_x$ and $EI\,w^{IV} = p_z$ this is true, since the homogeneous solutions

$$u_h(x) = c_1 + c_2\,x \tag{4.39a}$$

$$w_h(x) = c_1 + c_2\,x + c_3\,x^2 + c_4\,x^3 \tag{4.39b}$$

lie in $\mathcal{V}_h$, see Fig. 4.2, but if a bar must work against some friction (c)