Chapter 1
Foundation

1.1 Introduction

We start with the variational principles of statics and Betti's theorem,

- the principle of virtual displacements
- the principle of conservation of energy
- the principle of virtual forces
- Betti's theorem

Since they form the core of modern structural analysis. As diverse as these principles and theorems are, technically they are all based on one equation: Integration by parts.

1.1.1 Integration by Parts

The integration by parts formula

$$\int_0^l u'\,\delta u\,dx = [u\,\delta u]_0^l - \int_0^l u\,\delta u'\,dx\,, \tag{1.1}$$

derived from

$$\int_0^l (u\,\delta u)'\,dx = \int_0^l (u'\,\delta u + u\,\delta u')\,dx = [u\,\delta u]_0^l\,, \tag{1.2}$$

can be written as a **"zero sum"**,

F. Hartmann and P. Jahn, *Statics and Influence Functions*,
Springer Series in Solid and Structural Mechanics 13,
https://doi.org/10.1007/978-3-030-55889-5_1

$$\mathscr{I}(u, \delta u) = \int_0^l u' \, \delta u \, dx - [u \, \delta u]_0^l + \int_0^l u \, \delta u' \, dx = 0 \,, \tag{1.3}$$

since this invariant form better expresses the built-in **duality**: The equation is zero for ***all pairs*** of functions in $C^1(0, l)$, like $u = \sin(x)$ and $\delta u = \cos(x)$.

1.1.2 Principle of Virtual Displacements

Zero sums are easy to deal with. When two forces $\pm f$ pull at the two ends of a bar as in Fig. 1.1a, *the zero sum* of the two forces,

$$-f + f = 0 \,, \tag{1.4}$$

can be multiplied with any number δu without changing the mathematics

$$\delta u \cdot (-f + f) = -\delta u \cdot f + \delta u \cdot f = 0 \,, \tag{1.5}$$

which means we can slide the bar forwards and backwards on the table and each time the work done by the two forces is zero. This is an application of the ***principle of virtual displacements***.

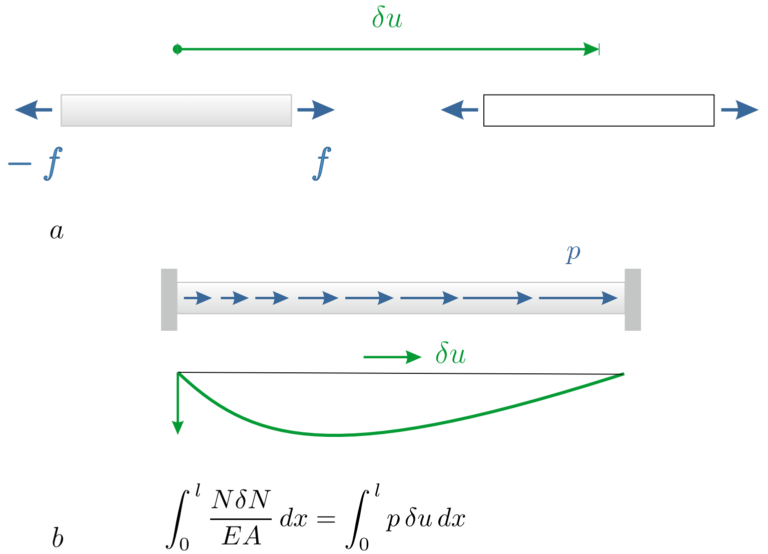


Fig. 1.1 Virtual displacement, **a** of an unrestrained rigid bar, and **b** an elastic bar fixed at both ends. For each admissible δu the two integrals have the same value

Granted, the logic is remarkably simple. If an equation is zero

$$Eq = 0\,, \tag{1.6}$$

the product of the equation with any number δu is zero as well

$$\delta u \cdot Eq = 0\,, \tag{1.7}$$

and this holds also true for functions, see Fig. 1.1b, since if $u(x)$ satisfies the differential equation

$$-EA\,u''(x) - p(x) = 0 \qquad 0 < x < l\,, \tag{1.8}$$

then

$$\int_0^l (-EA\,u'' - p)\,\delta u\,dx = 0\,, \tag{1.9}$$

or after integration by parts, if $\delta u(0) = \delta u(l) = 0$,

$$\int_0^l \frac{N\,\delta N}{EA}\,dx = \int_0^l p\,\delta u\,dx \tag{1.10}$$

where $N = EA\,u'$ is the normal force.

1.1.3 Betti's Theorem

When two numbers u_1 and u_2 solve the two **"twin"** equations (the 3 makes them twins)

$$3 \cdot u_1 = 12 \qquad 3 \cdot u_2 = 18\,, \tag{1.11}$$

and we multiply each twin with the other solution,

$$u_2 \cdot 3 \cdot u_1 = 12 \cdot u_2 \qquad u_1 \cdot 3 \cdot u_2 = 18 \cdot u_1\,, \tag{1.12}$$

then the left sides are the same, and so also the right sides must be the same, see Fig. 1.2,

$$W_{1,2} = 12 \cdot x_2 = 18 \cdot x_1 = W_{2,1}\,. \tag{1.13}$$

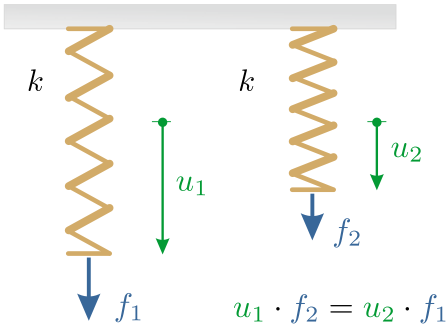


Fig. 1.2 Two springs and application of *Betti's theorem*, stiffness $k = 3$

This is *Betti's theorem*: *The reciprocal external work of two equilibrium states is the same*. It is a universal property of self-adjoint operators and symmetric matrices. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the nodal displacements of a truss under two different loads

$$\boldsymbol{K}\boldsymbol{u}_1 = \boldsymbol{f}_1 \qquad \boldsymbol{K}\boldsymbol{u}_2 = \boldsymbol{f}_2\,, \tag{1.14}$$

the left sides can be made the same

$$\boldsymbol{u}_2^T\boldsymbol{K}\boldsymbol{u}_1 = \boldsymbol{u}_2^T\boldsymbol{f}_1 \qquad \boldsymbol{u}_1^T\boldsymbol{K}\boldsymbol{u}_2 = \boldsymbol{u}_1^T\boldsymbol{f}_2\,, \tag{1.15}$$

and so also the right sides must be the same

$$\boldsymbol{u}_2^T\boldsymbol{f}_1 = \boldsymbol{u}_1^T\boldsymbol{f}_2\,. \tag{1.16}$$

1.1.4 Influence Functions

To solve the equation

$$3 \cdot x = 12 \tag{1.17}$$

we divide the right side by the number 3, or—as we could say as well—we multiply the right side with the inverse $g = 1/3$, the response to a "point load", a "Dirac delta",

$$3 \cdot g = 1\,, \tag{1.18}$$

since this equation implies that

$$x = g \cdot 12 = \frac{1}{3} \cdot 12 = 4 \tag{1.19}$$

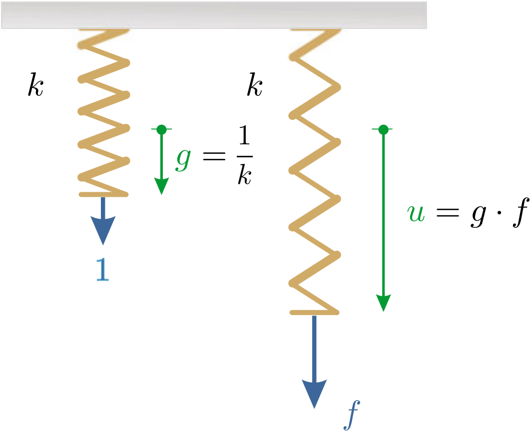


Fig. 1.3 If a force $f = 1$ stretches the spring by $g = 1/k$ units, a force f will stretch the spring by $u = f \cdot g$ units

is the solution to (1.17), see Fig. 1.3. *This is the technique of influence functions or Green's functions*.

To calculate the nodal displacement u_i of a truss in this way, we apply a single force $f_i = 1$ at the node, determine the corresponding nodal displacements of the truss, the vector $\boldsymbol{g}_i$,

$$\boldsymbol{K}\,\boldsymbol{g}_i = \boldsymbol{e}_i \qquad (i\text{-th unit vector})\,, \tag{1.20}$$

and form the scalar product of $\boldsymbol{g}_i$ and $\boldsymbol{f}$

$$u_i = \boldsymbol{e}_i^T\,\boldsymbol{u} = \boldsymbol{e}_i^T\,\boldsymbol{K}^{-1}\,\boldsymbol{f} = \boldsymbol{g}_i^T\,\boldsymbol{f}\,. \tag{1.21}$$

In a beam, instead, we would place a **single force** $P = 1$ at the **source point** x, determine the response $G(y, x)$ of the beam, see Fig. 1.4, and integrate

$$w(x) = \int_0^l G(y, x)\,p(y)\,dy\,. \tag{1.22}$$

1.1.5 Identities

In statics we solve scalar equations

$$k\,u = f\,, \tag{1.23}$$

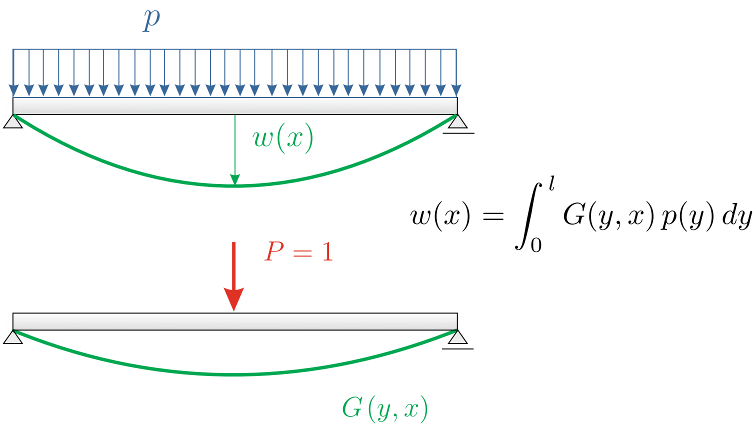


Fig. 1.4 Beam and influence function for the deflection at the midpoint

or systems of equations

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}\,, \tag{1.24}$$

or differential equations

$$E I\, w^{IV}(x) = p(x)\,. \tag{1.25}$$

To each of the operators on the left belongs a simple identity

$$\mathscr{B}(u, \delta u) = \delta u\, k\, u - u\, k\, \delta u = 0 \tag{1.26}$$

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{\delta u}) = \boldsymbol{\delta u}^T\, \boldsymbol{K}\, \boldsymbol{u} - \boldsymbol{u}^T\, \boldsymbol{K}\, \boldsymbol{\delta u} = 0 \tag{1.27}$$

$$\mathscr{G}(w, \delta w) = \int_0^l E I\, w^{IV}\, \delta w\, dx + [V\, \delta w - M\, \delta w']_0^l - \int_0^l \frac{M\, \delta M}{E I}\, dx = 0\,. \tag{1.28}$$

Only this last identity requires a warning, since it is based on integration by parts, and so the functions w and δw must be from $C^4(0, l)$ and $C^2(0, l)$ respectively for it to be true, since in (1.28) we integrate two times.

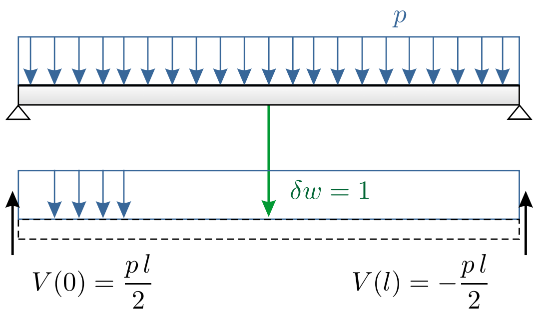


Fig. 1.5 Check of the equilibrium of the beam with Green's first identity by applying the virtual displacement $\delta w = 1$, $\mathscr{G}(w, 1) = p \cdot l + V(l) - V(0) = 0$

The energy principles of statics are based on Green's first and second identity (integration by parts)

This is why the essential formulations of statics and mechanics are ***dual formulations***, are "stereo", not "mono". The prime mathematical operation in mechanics is the **scalar product**. The ubiquitous *for all* δu of the variational principles has its root in the scalar product, see Fig. 1.5.

1.2 Green's Identities

To continue in a more systematic and orderly fashion we will list in the following the major differential equations of frame analysis, and formulate the associated identities as for example the identity of a bar

$$\int_0^l -EA\,u''\,\delta u\,dx = [(-EA\,u')\,\delta u]_0^l - \int_0^l -EA\,u'\,\delta u'\,dx\,. \tag{1.29}$$

The principal tool, integration by parts, is like climbing stairs ($\delta u = 1$)

$$\int_a^b f'(x)\,dx = f(b) - f(a)\,, \tag{1.30}$$

since with each step dx in horizontal direction the gain in height is $df = f'(x)\,dx$ and so at the end the total is $f(b) - f(a)$, see Fig. 1.6.

The staircase formula (1.30) is the ***fundamental theorem of calculus***. It implies that the integral of the normal force $N(x) = EA\,u'(x)$ in a bar, fixed at both ends, is zero, see Fig. 1.7a,

$$\int_0^l EA\,u'(x)\,dx = [EA\,u]_0^l = EA\,(u(l) - u(0)) = 0\,, \tag{1.31}$$

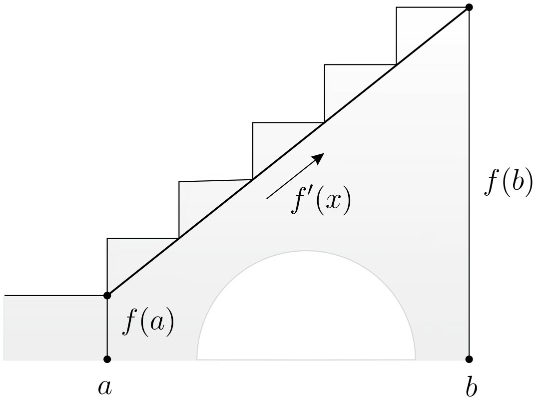


Fig. 1.6 On climbing stairs we sense the fundamental theorem of differential and integral calculus

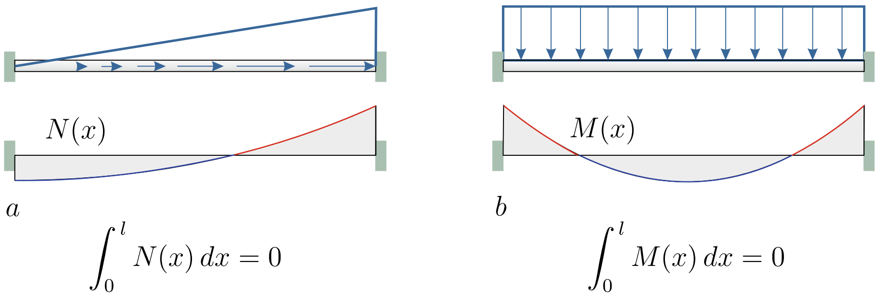


Fig. 1.7 The integrals of the normal force N and the bending moment M are zero

and the mean value of the bending moment $M(x) = -EI\,w''(x)$ in a beam, clamped on both sides, see Fig. 1.7b, is zero as well

$$\int_0^l -EI\,w''(x)\,dx = -EI\,(w'(l) - w'(0)) = 0\,. \tag{1.32}$$

For **partial derivatives** integration by parts reads

$$\int_\Omega u_{,i}\;\delta v\,d\Omega = \int_\Gamma u\,n_i\,\delta v\,ds - \int_\Omega u\,\delta v_{,i}\;d\Omega\,, \tag{1.33}$$

where Γ is the edge of the domain Ω, n_i is the i-th component of the normal vector $\boldsymbol{n}$ (length $|\boldsymbol{n}| = 1$) on Γ and $u_{,i} = \partial u/\partial x_i$ is the derivative with respect to x_i. If we let $\delta v = 1$ Eq. (1.33) becomes

$$\int_{\Omega} u_{,i} \, d\Omega = \int_{\Gamma} u \, n_i \, ds \,, \tag{1.34}$$

and so, if a plate Ω is fixed along its edge Γ, $u_x = u_y = 0$, the mean of the stress

$$\sigma_{xx} = E\,(\varepsilon_{xx} + \nu\,\varepsilon_{yy}) = E\,(u_{x,x} + \nu\,u_{y,y}) \tag{1.35}$$

(and of σ_{yy} as well) is zero, since

$$\int_{\Omega} E\,(u_{x,x} + \nu\,u_{y,y})\,d\Omega = \int_{\Gamma} E\,(u_x\,n_x + \nu\,u_y\,n_y)\,ds = 0\,. \tag{1.36}$$

1.2.1 Longitudinal Displacement $u(x)$ of a Bar

$$-EA\,u''(x) = p(x) \tag{1.37}$$

$$\mathscr{G}(u, \delta u) = \underbrace{\int_0^l -EA\,u''(x)\,\delta u(x)\,dx + [N\,\delta u]_0^l}_{\text{external virt. work}} - \underbrace{\int_0^l \frac{N\,\delta N}{EA}\,dx}_{\text{internal virt. work}} = 0\,, \tag{1.38}$$

where $N = EA\,u'$ is the normal force, see Fig. 1.8.

If $EA(x)$ is a function of x, the differential equation of the bar is $-(EA(x)\,u')' = p(x)$ and then

$$\int_0^l -(EA(x)\,u')'\,\delta u\,dx = [(-EA(x)\,u')\,\delta u]_0^l - \int_0^l -EA(x)\,u'\,\delta u'\,dx \tag{1.39}$$

which repeats the identity (1.38), because the definition of $N = EA(x)\,u'$ does not change

$$\mathscr{G}(u, \delta u) = \int_0^l -(EA(x)\,u')'\,\delta u(x)\,dx + [N\,\delta u]_0^l - \int_0^l \frac{N\,\delta N}{EA}\,dx = 0\,. \tag{1.40}$$

Since $-N' = p$ is the same as $-(EA(x)\,u')' = p$, we can also write

$$\int_0^l -N'\,\delta u\,dx = [N\,\delta u]_0^l - \int_0^l -N\,\delta u'\,dx\,. \tag{1.41}$$

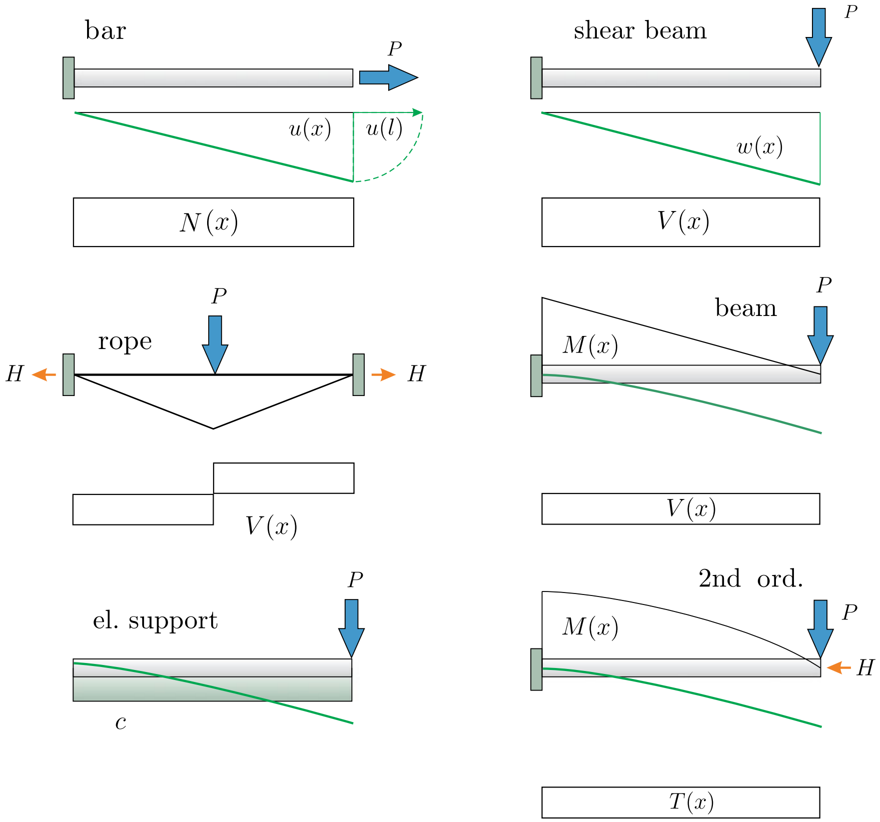


Fig. 1.8 Structural elements

When the longitudinal displacement must work against some friction (c),

$$-EA\,u''(x) + c\,u(x) = p(x)\,, \tag{1.42}$$

the identity reads

$$\mathscr{G}(u, \delta u) = \underbrace{\int_0^l (-EA\,u''(x) + c\,u(x))\,\delta u(x)\,dx + [N\,\delta u]_0^l}_{\delta W_e} - \underbrace{\int_0^l (\frac{N\,\delta N}{EA} + c\,u\,\delta u)\,dx}_{\delta W_i} = 0\,. \tag{1.43}$$

1.2.2 Shear Deformation $w_S(x)$ of a Beam

$$-GA\,w_s''(x) = p(x) \tag{1.44}$$

$$\mathscr{G}(w_s, \delta w_s) = \underbrace{\int_0^l -GA\,w_s''(x)\,\delta w_s(x)\,dx + [V\,\delta w_s]_0^l}_{\delta W_e} - \underbrace{\int_0^l \frac{V\,\delta V}{GA}\,dx}_{\delta W_i} = 0\,, \tag{1.45}$$

with $V = GA\,w_s'$ as the shear force, and GA as the shear modulus.

When the beam sits on an elastic foundation (c),

$$-GA\,w_s''(x) + c\,w_s(x) = p(x)\,, \tag{1.46}$$

the identity has the form

$$\mathscr{G}(w_s, \delta w_s) = \underbrace{\int_0^l (-GA\,w_s''(x) + c\,w_s(x))\,\delta w_s(x)\,dx + [V\,\delta w_s]_0^l}_{\delta W_e}$$

$$- \underbrace{\int_0^l (\frac{V\,\delta V}{GA} + c\,w_s\,\delta w_s)dx}_{\delta W_i} = 0\,. \tag{1.47}$$

1.2.3 Deflection w of a Rope

$$-H\,w''(x) = p(x) \qquad H = \text{horizontal prestress in the rope} \tag{1.48}$$

with $V(x) = H\,w'(x)$ as the shear force in the rope

$$\mathscr{G}(w, \delta w) = \underbrace{\int_0^l -H\,w''(x)\,\delta w(x)\,dx + [V\,\delta w]_0^l}_{\delta W_e} - \underbrace{\int_0^l \frac{V\,\delta V}{H}\,dx}_{\delta W_i} = 0\,. \tag{1.49}$$

1.2.4 Deflection w of a Beam

$$EI\, w^{IV}(x) = p(x) \tag{1.50}$$

$$\mathscr{G}(w, \delta w) = \underbrace{\int_0^l EI\, w^{IV}(x)\, \delta w\, dx + [V\, \delta w - M\, \delta w']_0^l}_{\delta W_e} - \underbrace{\int_0^l \frac{M\, \delta M}{EI}\, dx}_{\delta W_i} = 0\,, \tag{1.51}$$

with $M(x) = -EI\, w''(x)$ and $V(x) = -EI\, w'''(x)$.

If $EI(x)$ is a function of x, the beam equation is $(EI(x)\, w'')'' = p(x)$, and so

$$\begin{aligned}\int_0^l (EI(x)\, w'')''\, \delta w\, dx &= [(EI(x)\, w'')'\, \delta w - EI(x)\, w''\, \delta w']_0^l \\ &\quad + \int_0^l EI(x)\, w''\, \delta w''\, dx\,,\end{aligned} \tag{1.52}$$

where $M = -EI(x)\, w''$ and $V = -(EI(x)\, w'')'$ and this means

$$\mathscr{G}(w, \delta w) = \int_0^l (EI(x)\, w'')''\, \delta w\, dx + [V\, \delta w - M\, \delta w']_0^l - \int_0^l \frac{M\, \delta M}{EI}\, dx = 0\,. \tag{1.53}$$

Here too we can, since $-M'' = p$ is the same as $(EI(x)w'')'' = p$, write

$$\int_0^l -M''\, \delta w\, dx = -[M'\, \delta w]_0^l + \int_0^l M'\, \delta w'\, dx\,. \tag{1.54}$$

1.2.5 Deflection w of a Beam, 2nd Order Theory

$$EI\, w^{IV}(x) + (D(x)\, w'(x))' = p_z(x) \qquad D(x) = P + \int_0^x p_x(y)\, dy \tag{1.55}$$

$$\begin{aligned}\mathscr{G}(w, \delta w) = &\underbrace{\int_0^l (EI\, w^{IV}(x) + (D(x)\, w'(x))')\, \delta w\, dx + [T\, \delta w - M\, \delta w']_0^l}_{\delta W_e} \\ &- \underbrace{\int_0^l \left(\frac{M\, \delta M}{EI} - D(x)\, w'(x)\, \delta w'(x)\right) dx}_{\delta W_i} = 0\end{aligned} \tag{1.56}$$

where T is the *transverse force*

$$T(x) = -EI\, w'''(x) - D(x)\, w'(x) = V(x) - D(x)\, w'(x)\,. \tag{1.57}$$

The D is the compressive force and $p_x(x)$ and $p_z(x)$ are distributed forces in axial direction and perpendicular to it.

1.2.6 Beam on an Elastic foundation

$$EI\, w^{IV}(x) + c\, w(x) = p(x) \tag{1.58}$$

and

$$\begin{aligned}\mathscr{G}(w, \delta w) &= \underbrace{\int_0^l (EI\, w^{IV}(x) + c\, w(x))\, \delta w(x)\, dx + [V\, \delta w - M\, \delta w']_0^l}_{\delta W_e} \\ &\quad - \underbrace{\int_0^l (\frac{M\, \delta M}{EI} + c\, w(x)\, \delta w(x))\, dx}_{\delta W_i} = 0\,. \end{aligned} \tag{1.59}$$

1.2.7 Tensile Chord Bridge

Imagine a beam through which runs a prestressed rope (tendon), so that the beam and the rope jointly carry the distributed load p

$$EI\, w^{IV}(x) - H\, w''(x) = p(x) \qquad H = \text{prestress}\,, \tag{1.60}$$

and Green's first identity reads

$$\begin{aligned}\mathscr{G}(w, \delta w) &= \underbrace{\int_0^l (EI\, w^{IV}(x) - H\, w''(x))\, \delta w(x)\, dx + [V\, \delta w - M\, \delta w']_0^l}_{\delta W_e} \\ &\quad - \underbrace{\int_0^l (\frac{M\, \delta M}{EI} + H\, w'(x)\, \delta w'(x))\, dx}_{\delta W_i} = 0\,, \end{aligned} \tag{1.61}$$

with $V = -EI\, w'''(x) + H\, w'(x)$.

1.2.8 Torsion

The differential equation of ***St. Venant's torsion***

$$-G\,I_T\,\vartheta'' = m_x \tag{1.62}$$

$$\mathscr{G}(\vartheta, \delta\vartheta) = \underbrace{\int_0^l -G\,I_T\,\vartheta''(x)\,\delta\vartheta(x)\,dx + [M_T\,\delta\vartheta]_0^l}_{\delta W_e} - \underbrace{\int_0^l \frac{M_T\,\delta M_T}{G\,I_T}\,dx}_{\delta W_i} = 0\,, \tag{1.63}$$

and ***warping torsion***

$$E I_\omega\,\vartheta^{IV} - G\,I_T\,\vartheta'' = m_x \tag{1.64}$$

$$\mathscr{G}(\vartheta, \delta\vartheta) = \underbrace{\int_0^l (E I_\omega\,\vartheta^{IV}(x) - G\,I_T\,\vartheta''(x))\,\delta\vartheta(x)\,dx + [M_T\,\delta\vartheta - M_\omega\,\delta\vartheta']_0^l}_{\delta W_e}$$
$$-\underbrace{\int_0^l (\frac{M_\omega\,\delta M_\omega}{E I_\omega} + G\,I_T\,\vartheta'(x)\,\delta\vartheta'(x))\,dx}_{\delta W_i} = 0\,, \tag{1.65}$$

with

$$M_\omega = -E I_\omega\,\vartheta''(x) \qquad M_T = -E I_\omega\,\vartheta'''(x) + G\,I_T\,\vartheta'(x) \tag{1.66}$$

repeat the patterns from above.

1.3 Variational Principles of Structural Analysis

In all identities, as for example the identity of a rope,

$$\mathscr{G}(w, \delta w) = \int_0^l -H\,w''(x)\,\delta w(x)\,dx + [V\,\delta w]_0^l - \int_0^l \frac{V\,\delta V}{H}\,dx = 0\,, \tag{1.67}$$

we equate the external virtual work with the internal strain energy. So, each term is an energy, is the scalar product of a **force** [N] and a **displacement** [m] and the bottom line is: The sum is always zero, the account is always balanced. This simple observation is the key to the variational and energy principles of statics.

Principle of Virtual Displacements

$$\boxed{\mathscr{G}(w, \delta w) = \delta W_e - \delta W_i = 0} \tag{1.68}$$

Conservation of Energy

If the second entry is identical with the first, $\delta w = w$, the identities formulate the *principle of conservation of energy*

$$\boxed{\frac{1}{2}\,\mathscr{G}(w, w) = W_e - W_i = 0} \tag{1.69}$$

The external eigenwork is stored as internal energy (this needs the factor $1/2$).

Principle of Virtual Forces

If the test function δw^* takes the first spot, and—as is tradition—is then written with an asterisk, it is the *principle of virtual forces*

$$\boxed{\mathscr{G}(\delta w^*, w) = \delta W_e^* - \delta W_i^* = 0} \tag{1.70}$$

Betti's Theorem

When we formulate Green's first identity twice, but switch the positions of w and $\hat{w}$ in the second round, and subtract the two equations, $0 - 0 = 0$, we arrive at Green's second identity

$$\mathscr{B}(w,\, \hat{w}) = \mathscr{G}(w, \hat{w}) - \mathscr{G}(\hat{w}, w) = \underbrace{\int_0^l EI\, w^{IV}(x)\, \hat{w}(x)\, dx + [V\, \hat{w} - M\, \hat{w}']_0^l}_{W_{1,2}}$$
$$- \underbrace{[w\, \hat{V} - w' \hat{M}]_0^l - \int_0^l w(x)\, EI\, \hat{w}^{IV}(x)\, dx}_{W_{2,1}} = 0\,, \tag{1.71}$$

or *Betti's theorem*: The reciprocal external work of two deflections w and $\hat{w}$ is the same.

$$\boxed{\mathscr{B}(w, \hat{w}) = W_{1,2} - W_{2,1} = 0} \tag{1.72}$$

Principle of Minimum Potential Energy

The **potential energy** of a hinged beam—in classical and modern notation side by side—is the expression

$$\Pi(w) = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx - \int_0^l p(x)\,w(x)\,dx = \frac{1}{2}\,a(w,w) - (p,w)$$
$$= \frac{1}{2}\,a(w,w) - \frac{1}{2}\,(p,w) - \frac{1}{2}\,(p,w)\,. \tag{1.73}$$

If w is the deflection of the beam, $EI\,w^{IV} = p$, Green's first identity implies $\mathcal{G}(w,w) = a(w,w) - (p,w) = 0$, and $\Pi(w)$ reduces to

$$\boxed{\Pi(w) = -\frac{1}{2}\,(p,w) = \text{half the eigenwork} \times (-1)} \tag{1.74}$$

which means: The potential energy in the equilibrium position is negative, since ***eigenwork*** (p,w) is always positive.

If we add an admissible virtual displacement δw, i.e. $\delta w(0) = \delta w(l) = 0$, to the equilibrium position w, the potential energy will increase

$$\Pi(w + \delta w) = \Pi(w) + \underbrace{\mathcal{G}(w, \delta w)}_{=0} + \underbrace{a(\delta w, \delta w)}_{>0}\,, \tag{1.75}$$

and so $\Pi(w)$ must be **the deepest point**.

1.4 Zero Sums

Green's first identity resembles the game played by the **desert wind** ($= \delta u$) with the dried-out ***tumbleweed*** ($= u$), see Fig. 1.9. No matter how strong the wind blows and how big the capers are, at the end the result always comes out zero, $\mathcal{G}(u, \delta u) = 0$.

If we read the identity as a **variational statement**

$$\mathcal{G}(u, \delta u) = 0 \qquad \text{for all } \delta u\,, \tag{1.76}$$

it reminds of the **path-independence** of the work integral of a point mass m in the gravitational field of the Earth, see Fig. 1.10. Near the Earth's surface the potential energy has the form $\Pi = m \cdot g \cdot y$, and if the point mass m moves on a closed path $\mathcal{C} = \{x(s), y(s)\}^T$ its total work is zero[1]

[1] Int. by parts and $y(0) = y(L)$ with $L =$ length of the path.

Fig. 1.9 Tumbleweed

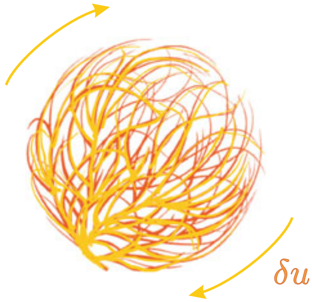
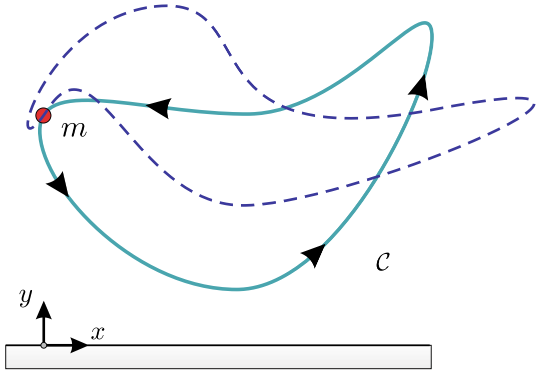


Fig. 1.10 Closed path



$$\int_{\mathcal{C}} \cdot \nabla \Pi \bullet \boldsymbol{ds} = m \cdot g \int_0^L \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x' \\ y' \end{bmatrix} ds = m \cdot g \int_0^L y' \, ds$$
$$= m \cdot g \cdot (y(L) - y(0)) = 0\,, \tag{1.77}$$

regardless of the shape of the curve $\mathcal{C}$—the path δu so to speak.

It all starts with the **scalar product**[2] **of two conjugated quantities**, of a force and a displacement,

$$\int_0^l -E A\, u'' \,\delta u \, dx = \textit{force} \times \textit{displacement} \tag{1.78}$$

and integration by parts

$$\mathscr{I}(u, \delta u) = \int_0^l u' \,\delta u \, dx - [u\,\delta u]_0^l + \int_0^l u\,\delta u' \, dx = 0\,, \tag{1.79}$$

[2]The superposition of two functions is called an L_2-scalar product.